Deformation of two body quantum Calogero-Moser-Sutherland models

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Abstract

The possibility of deformation of two body quantum Calogero-Moser-Sutherland models is studied. Obtained are some necessary conditions for the singular locus of the potential function. Such locus is determined if it consists of two, three or four lines. Furthermore, a new deformation of elliptic B_2 type Calogero-Moser-Sutherland model is explicitly constructed.

1 Introduction

A Schrödinger operator

$$L := -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + R(x)$$

is called completely integrable if there exist n algebraically independent differential operators $P_1 = L, P_2, \ldots, P_n$ which commute each other. Let (Σ, W) be a pair of a root system and its Weyl group. The n-body Calogero-Moser-Sutherland (CMS) operator

$$L = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha \in \Sigma^+} m_\alpha(m_\alpha + 1)|\alpha|^2 u(\langle \alpha, x \rangle), \tag{1.1}$$

with

$$u_\alpha(t) = \begin{cases} 1/t^2 \quad \text{(rational case)}, \\ \omega^2/\sin^2 \omega t, \;\; \omega^2/\sinh^2 \omega t \quad \text{(trigonometric case)}, \\ \wp(t) \quad \text{(elliptic case)}, \end{cases}$$

$$m_{w\alpha} = m_\alpha \quad (\alpha \in \Sigma, w \in W),$$

is an example of completely integrable operator. Here, $\wp(t)$ is the Weierstrass $\wp$ function. The constants m_α are called the *coupling constants*.

Obviously, these potential functions possess inverse square singularities along the walls of Weyl chambers. As a generalisation of CMS operator, let us consider a Schrödinger operator

$$L = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + R(x), \quad \text{with} \quad R(x) = \sum_{\alpha \in \mathcal{H}} \frac{C_\alpha}{\langle \alpha, x \rangle^2} + \widetilde{R}(x). \tag{1.2}$$

Here, $\mathcal{H}$ is a finite set of mutually non-parallel vectors in $\boldsymbol{R}^n$, C_α are non-zero constants and $\widetilde{R}(x)$ is real analytic at $x = 0$. We call $\mathcal{H}$ the *singular locus* of L or the singular locus of $R(x)$. Note

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that we do not assume the symmetry of either $R(x)$ or P, nor do we assume $\mathcal{H}$ to be a subset of a root system.

In [4], the author investigated what kind of differential operator P commutes with L in (1.2). One of the main results of [4] is that, if $C_\alpha \notin \{m(m+1)|\alpha|^2; m \in \boldsymbol{Z}\}$ for any $\alpha \in \mathcal{H}$, then the principal symbol of P is invariant under the action of the group W generated by reflections r_α with respect to the hyperplanes $\langle \alpha, x\rangle = 0$ ($\alpha \in \mathcal{H}$). Therefore, if L possesses a non-trivial commutant, then W must be a finite reflection group and $\mathcal{H}$ must be a subset of the root system of this reflection group ([4, Theorem 4.4]).

On the other hand, if some of the coupling constants are one, i.e. $C_\alpha = 1 \cdot 2|\alpha|^2$ for some $\alpha \in \mathcal{H}$, it is known that there exist completely integrable Schrödinger operators like (1.1), but whose singular loci are not root systems but deformed ones [2, 5].

The final objective of this research is to classify such deformed completely integrable CMS type operators and to construct such operators explicitly. But, in this paper, we do not consider general cases, but restrict our interest to the rank two rational cases. Namely, we consider what kind of operator P commutes with

$$L = -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + \sum_{\alpha \in \mathcal{H}} \frac{C_\alpha}{\langle \alpha, x\rangle^2}, \qquad (\mathcal{H} \subset \boldsymbol{R}^2, C_\alpha \neq 0). \tag{1.3}$$

The reason to do so is as follows: If the operator L in (1.2) and a differential operator P commute, then, by "restricting" them to a two dimensional subspace, we obtain a two body completely integrable CMS type operator L', whose potential function is a rational function. For details, see §2. Therefore, two body rational integrable models are building blocks of general integrable models, and it is important to classify and construct them.

The first result of this paper is the relation between the order of P and the cardinality of $\mathcal{H}$.

Theorem 1.1 (Corollary 3.4) *Assume that the Schrödinger operator L in (1.3) has a non-trivial commutant P, whose principal symbol is constant with respect to x. Then the order of P is not less than the cardinality of $\mathcal{H}$.*

Next results are the conditions for the singular locus $\mathcal{H}$ and the constants C_α. For $\alpha = (\alpha_1, \alpha_2) \in \mathcal{H}$, let $\alpha^\perp = (-\alpha_2, \alpha_1)$.

Theorem 1.2 (Theorem 4.2, Theorem 5.2) *Assume that L and P satisfy the same condition as in Theorem 1.1. Then, for each $\alpha_0 \in \mathcal{H}$,*

$$\sum_{\beta \in \mathcal{H}, \beta \neq \alpha_0} \frac{\langle \alpha_0, \beta\rangle}{\langle \alpha_0^\perp, \beta\rangle^3} C_\beta = 0 \qquad \textit{and} \qquad (C_{\alpha_0} - 2|\alpha_0|^2) \sum_{\beta \in \mathcal{H}, \beta \neq \alpha_0} \frac{\langle \alpha_0, \beta\rangle |\beta|^2}{\langle \alpha_0^\perp, \beta\rangle^5} C_\beta = 0 \tag{1.4}$$

are satisfied.

In §§4, 6, we investigate what kind of $\mathcal{H}$ and C_α satisfy (1.4), when $\#\mathcal{H} = 2, 3$ or 4. Since $\mathcal{H}$ describes the singular locus of the potential function in (1.3), the norm of each vector in $\mathcal{H}$ is not essential. Actually, if you replace $\alpha \in \mathcal{H}$ and C_α with $k\alpha$ and $k^2 C_\alpha$ ($k \in \boldsymbol{R}^\times$) respectively, the operator L and the conditions (1.4) are unchanged. Therefore, we consider two singular loci $\mathcal{H}$ and $\mathcal{H}'$ to be equivalent if each vector in $\mathcal{H}'$ is a non-zero multiple of a vector in $\mathcal{H}$. Moreover, we also consider $\mathcal{H}$ and $\mathcal{H}'$ to be equivalent if $\mathcal{H}' = \{g\alpha; \alpha \in \mathcal{H}\}$ for some $g \in O(2)$.

Theorem 1.3 (Corollary 4.3, Theorem 6.3, Theorem 6.5) *(1) If $\#\mathcal{H} = 2$, then the two vectors in $\mathcal{H}$ cross at right angles. Therefore, the singular locus is of type $A_1 \times A_1$.*

(2) If $\#\mathcal{H} = 3$, then $\mathcal{H} = \{e_1, \pm ae_1 + e_2\}$ for some $a \neq 0$. Moreover, if $\mathcal{H}$ is not a positive system of A_2 type root system, then the coupling constants for $\pm ae_1 + e_2$ must be one and there is no other completely integrable model than the one constructed in [2].

(3) If $\#\mathcal{H} = 4$, then $\mathcal{H} = \{e_1, e_2, \pm ae_1 + e_2\}$ for some $a \neq 0$.

As stated above, deformation of CMS operators is known if some of the coupling constants are one. On the other hand, Theorem 1.2 implies that there may be other deformation of a CMS operator even if no coupling constant is one. In §7, we present an example of a new deformation of the B_2 type CMS operator. The result is as follows.

Theorem 1.4 (Theorem 7.2) *Let L be the Schrödinger operator defined by*

$$\begin{aligned} L &= -\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + u_1(2ax_1) + u_2(2x_2) + u_+(ax_1 + x_2) + u_-(-ax_1 + x_2), \\ &\quad u_1(t) = \frac{3}{4}(a^2+1)(3a^{-2}-1)\wp(t), \qquad u_2(t) = \frac{3}{4}(a^2+1)(3a^2-1)\wp(t), \\ &\quad u_+(t) = u_-(t) = 2 \cdot 3(a^2+1)\wp(t). \end{aligned}$$

Then, there exists a sixth order commutant P of L, whose principal symbol is

$$a(4-a^2)\xi_1^6 + 5a\xi_1^4\xi_2^2 + 5a^{-1}\xi_1^2\xi_2^4 + a^{-1}(4-a^{-2})\xi_2^6.$$

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2 Rank two reduction

To begin, we introduce some notation. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\boldsymbol{R}^n$ and $x = (x_1, \dots, x_n)$ be the corresponding coordinates. For simplicity, denote by ∂_{x_i} the partial differential $\partial/\partial x_i$ and define $\partial_x = (\partial_{x_1}, \dots, \partial_{x_n})$. An m_0-th order differential operator P is expressed as

$$P = \sum_{k=0}^{m_0} P_k, \qquad P_k = \sum_{|p|=m_0-k} a_p(x)\partial_x^p,$$

where $p = (p_1, \dots, p_n) \in \boldsymbol{N}^n$ is a multi-index, and $|p|$ is the length $\sum_i p_i$ of p. Corresponding to this operator, we introduce

$$\widetilde{P}_k = \sum_{|p|=m_0-k} a_p(x)\xi^p \qquad (\xi = (\xi_1, \dots, \xi_n)),$$

and call it the *symbol* of P_k. In particular, $\widetilde{P}_0$ is called the *principal symbol* of P.

Let $\langle u, v\rangle$ be the standard inner product on $\boldsymbol{R}^n$, and let $|v|$ be the norm of v. We also use the symbol $\langle\ ,\ \rangle$ to other couplings. For example, $\langle \partial_x, \partial_\xi\rangle = \sum_{i=1}^n \partial_{x_i}\partial_{\xi_i}$. For notational convenience, let

$$x_\alpha := \langle \alpha, x\rangle, \qquad \xi_\alpha := \langle \alpha, \xi\rangle, \qquad \partial_{x,\alpha} := \langle \alpha, \partial_x\rangle, \qquad \partial_{\xi,\alpha} := \langle \alpha, \partial_\xi\rangle.$$

Assume that the operator L in (1.2) commutes with P, whose principal symbol $\tilde{P}_0$ is constant with respect to x. Then, by rank one reduction, we have the following results.

Lemma 2.1 *([4, Lemma 2.1]) For any $\alpha \in \mathcal{H}$, P is regular singular along the hyperplane $x_\alpha = 0$, i.e. $x_\alpha^k \widetilde{P}_k$ is analytic at $x_\alpha = 0$.*

Put $x' = (x'_1, x'_2, x'_3, \ldots, x'_n) = (\varepsilon^{-1}x_1, \varepsilon^{-1}x_2, x_3, \ldots, x_n)$ and consider the Laurent expansion of L and P as meromorphic functions of ε. By Lemma 2.1 and $\partial_x = (\varepsilon^{-1}\partial_{x'_1}, \varepsilon^{-1}\partial_{x'_2}, \partial_{x'_3}, \ldots, \partial_{x'_n},)$, we have

$$L = \sum_{i=-2}^{\infty} \varepsilon^i L(i), \qquad L(-2) = -(\partial_{x'_1}^2 + \partial_{x'_2}^2) + \sum_{\alpha \in \mathcal{H} \cap (\boldsymbol{R}e_1 + \boldsymbol{R}e_2)} \frac{C_\alpha}{{x'_\alpha}^2}$$

$$P = \sum_{i=-m_0}^{\infty} \varepsilon^i P(i), \qquad P(-m_0) \text{ is a differential operator on } \boldsymbol{R}e_1 + \boldsymbol{R}e_2.$$

Here, $L(i)$ and $P(i)$ are differential operators. Moreover, by Lemma 2.1, the principal symbol of $P(-m_0)$ is constant with respect to x. This expansion implies that if $[L, P] = 0$, we have $[L(-2), P(-m_0)] = 0$. In other words, we obtain a two body rational CMS type completely integrable system.

Therefore, we restrict our interest to this case. Namely, let $n = 2$ and define afresh L and P by

$$L = -(\partial_{x_1}^2 + \partial_{x_2}^2) + R(x), \quad R(x) = \sum_{\alpha \in \mathcal{H}} u_\alpha(x_\alpha), \quad u_\alpha(t) := \frac{C_\alpha}{t^2},$$

$$P = \sum_{k=0}^{m_0} P_k, \quad P_k = \sum_{p \in \boldsymbol{N}^2, |p| = m_0 - k} a_p(x) \partial_x^p,$$

where $\mathcal{H} \subset \boldsymbol{R}^2$, $C_\alpha \neq 0$ for any $\alpha \in \mathcal{H}$, and $a_p(x)$ is a homogeneous rational function of degree $-k$ if $|p| = k$. Hereafter, we will seek conditions for $\mathcal{H}$, C_α or P so that L and P commute. For notational convenience, we will abbreviate $u_\alpha^{(k)}(x_\alpha)$ to $u_\alpha^{(k)}$.

3 Order condition for P

By Leibniz rule, we have

$$[P_k, R(x)]^{\sim} = \sum_{j=1}^{m_0-k} \frac{1}{j!} \sum_{\alpha \in \mathcal{H}} u_\alpha^{(j)} \partial_{\xi,\alpha}^j \widetilde{P}_k.$$

Therefore, we have the following lemma.

Lemma 3.1 *The condition $[L, P] = 0$ is equivalent to*

$$2\langle \xi, \partial_x \rangle \widetilde{P}_k + \Delta \widetilde{P}_{k-1} + \sum_{j=1}^{k-1} \frac{1}{j!} \sum_{\alpha \in \mathcal{H}} u_\alpha^{(j)} \partial_{\xi,\alpha}^j \widetilde{P}_{k-j-1} = 0$$

for any $k = 0, \ldots, m_0$. Here, we set $\widetilde{P}_{-1} = \widetilde{P}_{m_0+1} = 0$, and we defined $\langle \xi, \partial_x \rangle := \xi_1 \partial_{x_1} + \xi_2 \partial_{x_2}$, $\Delta := \partial_{x_1}^2 + \partial_{x_2}^2$.

By Lemma 2.2 and Lemma 3.2 in [4], we can easily show the following proposition.

Proposition 3.2 *Choose $\alpha \in \mathcal{H}$ and express $\widetilde{P}_0$ as a polynomial in ξ_α, $\xi_{\alpha^\perp}$;*

$$\widetilde{P}_0 = \sum_{k=0}^{m_0} c_k \xi_\alpha^k \xi_{\alpha^\perp}^{m_0-k}.$$

(1) If $C_\alpha \neq m(m+1)|\alpha|^2$ for any $m \in \boldsymbol{Z}$, then $c_k = 0$ for all odd k.

(2) If $C_\alpha = m(m+1)|\alpha|^2$ for some $m \in \boldsymbol{Z}_{>0}$, then $c_1 = c_3 = \cdots = c_{2m-1} = 0$.

Especially, $\partial_{\xi,\alpha}\widetilde{P}_0|_{\xi_\alpha \to 0} = 0$ for any $\alpha \in \mathcal{H}$ since $C_\alpha \neq 0$.

Proposition 3.3 *Let D_θ be the differential operator $\xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2}$. If c_1 in the above proposition is 0, $D_\theta\widetilde{P}_0$ is divisible by ξ_α.*

Proof. For any $v = (v_1, v_2)$, $w = (w_1, w_2) \in \boldsymbol{R}^2$, we have

$$\begin{aligned} \xi_w \partial_{\xi,v} - \xi_v \partial_{\xi,w} &= (w_1\xi_1 + w_2\xi_2)(v_1\partial_{\xi_1} + v_2\partial_{\xi_2}) - (v_1\xi_1 + v_2\xi_2)(w_1\partial_{\xi_1} + w_2\partial_{\xi_2}) \\ &= (w_2v_1 - w_1v_2)(\xi_2\partial_{\xi_1} - \xi_1\partial_{\xi_2}) \\ &= \langle v^\perp, w\rangle D_\theta. \end{aligned} \tag{3.1}$$

Therefore,

$$\begin{aligned} D_\theta\widetilde{P}_0 &= \frac{1}{|\alpha|^2}(\xi_{\alpha^\perp}\partial_{\xi,\alpha} - \xi_\alpha\partial_{\xi,\alpha^\perp})\left(\sum_{k=0}^{m_0} c_k \xi_\alpha^k \xi_{\alpha^\perp}^{m_0-k}\right) \\ &= \sum_{k=2}^{m_0} k c_k \xi_\alpha^{k-1}\xi_{\alpha^\perp}^{m_0-k+1} - \sum_{k=0}^{m_0}(m_0-k)c_k\xi_\alpha^{k+1}\xi_{\alpha^\perp}^{m_0-k-1}, \end{aligned}$$

since $c_1 = 0$. The right hand side is divisible by ξ_α. □

Corollary 3.4 *If L and P commute, then $D_\theta\widetilde{P}_0$ is divisible by $\prod_{\alpha\in\mathcal{H}}\xi_\alpha$. Therefore, if $\widetilde{P}_0$ is not a polynomial in $\xi_1^2 + \xi_2^2$, the order of P is not less than the cardinality of $\mathcal{H}$.*

Proof. The first part is a direct consequence of Proposition 3.2, 3.3. Since $D_\theta\widetilde{P}_0 = 0$ is equivalent to $\widetilde{P}_0 \in \boldsymbol{C}[\xi_1^2 + \xi_2^2]$, the second assertion follows from the first one. □

4 Construction of $\widetilde{P}_2$, $\widetilde{P}_3$ and $\widetilde{P}_4$

For a differential operator $Q = \sum_p a_p(x)\partial_x^p$, let ${}^tQ$ be the formal adjoint operator $\sum_p(-\partial_x)^p \circ a_p(x)$ of Q. Since L is formally self-adjoint, if P commutes with L, so does ${}^tP$. Therefore, we may assume that P is formally (skew-)self-adjoint, that is, ${}^tP = (-1)^{\mathrm{ord}P}P$.

Lemma 4.1 *If P is formally (skew-)self-adjoint, then*

$$\widetilde{P}_{2k+1} = \frac{1}{2}\sum_{j=1}^{2k+1}\frac{(-1)^{j+1}}{j!}\langle\partial_x, \partial_\xi\rangle^j\widetilde{P}_{2k+1-j}.$$

Proof. By the Leibniz rule, we have

$$(-1)^{\mathrm{ord}P} \times {}^tP = \sum_{l=0}^{\mathrm{ord}P}(-1)^k\sum_{j=0}^{k}\frac{(-1)^j}{j!}\langle\partial_x, \partial_\xi\rangle^j\widetilde{P}_{l-j}.$$

The lemma is easily deduced from this equation. □

Since $\widetilde{P}_0$ is constant with respect to x, this lemma implies $\widetilde{P}_1 = 0$. By Lemma 3.1, $\widetilde{P}_2$ satisfies

$$2\langle \xi, \partial_x \rangle \widetilde{P}_2 + \sum_{\alpha \in \mathcal{H}} u'_\alpha \partial_{\xi,\alpha} \widetilde{P}_0 = 0 \quad \Leftrightarrow \quad \langle \xi, \partial_x \rangle \left(\widetilde{P}_2 + \frac{1}{2} \sum_{\alpha \in \mathcal{H}} u_\alpha \frac{\partial_{\xi,\alpha} \widetilde{P}_0}{\xi_\alpha} \right) = 0.$$

Note that Proposition 3.2 implies that $\partial_{\xi,\alpha} \widetilde{P}_0 / \xi_\alpha$ is a polynomial.

Let $\widetilde{Q} = \widetilde{P}_2 + (1/2) \sum_{\alpha \in \mathcal{H}} u_\alpha \partial_{\xi,\alpha} \widetilde{P}_0 / \xi_\alpha$. It is a polynomial in ξ of degree $m_0 - 2$ and its coefficients are homogeneous rational functions of degree -2. On the other hand, since $\widetilde{Q}$ satisfies $\langle \xi, \partial_x \rangle \widetilde{Q} = 0$, it is a function in ξ_1, ξ_2 and $x_2 \xi_1 - x_1 \xi_2$. From these conditions, we can conclude $\widetilde{Q} = 0$. Therefore,

$$\widetilde{P}_2 = -\frac{1}{2} \sum_{\alpha \in \mathcal{H}} u_\alpha \widetilde{P}_2^\alpha, \quad \widetilde{P}_2^\alpha := \frac{\partial_{\xi,\alpha} \widetilde{P}_0}{\xi_\alpha} \quad \text{and} \quad \widetilde{P}_3 = \frac{1}{2} \langle \partial_x, \partial_\xi \rangle \widetilde{P}_2 = -\frac{1}{4} \sum_{\alpha \in \mathcal{H}} u'_\alpha \partial_{\xi,\alpha} \widetilde{P}_2^\alpha. \tag{4.1}$$

By these formulae and Lemma 3.1, $\widetilde{P}_4$ satisfies the following equation:

$$\begin{aligned} \langle \xi, \partial_x \rangle \widetilde{P}_4 &= \frac{1}{8} \sum_{\alpha \in \mathcal{H}} |\alpha|^2 u_\alpha^{(3)} \partial_{\xi,\alpha} \widetilde{P}_2^\alpha + \frac{1}{4} \sum_{\alpha,\beta \in \mathcal{H}} u'_\alpha u_\beta \partial_{\xi,\alpha} \widetilde{P}_2^\beta - \frac{1}{12} \sum_{\alpha \in \mathcal{H}} u_\alpha^{(3)} \partial_{\xi,\alpha}^3 \widetilde{P}_0 \\ &= \langle \xi, \partial_x \rangle \sum_{\alpha \in \mathcal{H}} \left((u_\alpha^2 + |\alpha|^2 u''_\alpha) \frac{\partial_{\xi,\alpha} \widetilde{P}_2^\alpha}{8 \xi_\alpha} - u''_\alpha \frac{\partial_{\xi,\alpha}^3 \widetilde{P}_0}{12 \xi_\alpha} \right) + \frac{1}{4} \sum_{\substack{\alpha,\beta \in \mathcal{H} \\ \alpha \neq \beta}} u'_\alpha u_\beta \partial_{\xi,\alpha} \widetilde{P}_2^\beta. \end{aligned} \tag{4.2}$$

Now, since $u_\alpha(t) = C_\alpha / t^2$, we have $u_\alpha(t)^2 = C_\alpha u''_\alpha(t)/6$ and

$$\sum_{\alpha \in \mathcal{H}} \left((u_\alpha^2 + |\alpha|^2 u''_\alpha) \frac{\partial_{\xi,\alpha} \widetilde{P}_2^\alpha}{8 \xi_\alpha} - u''_\alpha \frac{\partial_{\xi,\alpha}^3 \widetilde{P}_0}{12 \xi_\alpha} \right) = \frac{1}{48} \sum_{\alpha \in \mathcal{H}} u''_\alpha \widetilde{P}_4^\alpha$$
$$\text{where } \widetilde{P}_4^\alpha = \frac{(C_\alpha + 6|\alpha|^2) \partial_{\xi,\alpha} \widetilde{P}_2^\alpha - 4 \partial_{\xi,\alpha}^3 \widetilde{P}_0}{\xi_\alpha}.$$

By Proposition 3.2, the coefficients c_1, c_3 in the expression $\widetilde{P}_0 = \sum_{k=0}^{m_0} c_k \xi_\alpha^k \xi_{\alpha^\perp}^{m_0 - k}$ satisfy $c_1 = c_3(C_\alpha - 2|\alpha|^2) = 0$. Therefore, $\widetilde{P}_4^\alpha$ is a polynomial in ξ, since

$$\begin{aligned} (C_\alpha + 6|\alpha|^2) \partial_{\xi,\alpha} \widetilde{P}_2^\alpha - 4 \partial_{\xi,\alpha}^3 \widetilde{P}_0 &= |\alpha|^4 \sum_{k=0}^{m_0} k(k-2) \{ C_\alpha + (10 - 4k) |\alpha|^2 \} c_k \xi_\alpha^{k-3} \xi_{\alpha^\perp}^{m_0 - k} \\ &= |\alpha|^4 \sum_{k=4}^{m_0} k(k-2) \{ C_\alpha + (10 - 4k) |\alpha|^2 \} c_k \xi_\alpha^{k-3} \xi_{\alpha^\perp}^{m_0 - k}, \end{aligned}$$

is divisible by ξ_α. Moreover, the last term in (4.2) is expressed as

$$\langle \xi, \partial_x \rangle F(x, \xi) = \frac{1}{4} \sum_{\substack{\alpha,\beta \in \mathcal{H} \\ \alpha \neq \beta}} u'_\alpha u_\beta \partial_{\xi,\alpha} \widetilde{P}_2^\beta, \quad F(x, \xi) := \widetilde{P}_4 - \frac{1}{48} \sum_{\alpha \in \mathcal{H}} u''_\alpha \widetilde{P}_4^\alpha. \tag{4.3}$$

Here, $F(x, \xi)$ is a polynomial in ξ and a meromorphic function in x with poles along $x_\alpha = 0$ of order at most two for each $\alpha \in \mathcal{H}$. Therefore, we have

$$\begin{aligned} &\lim_{x_{\alpha_0} \to 0} (x_{\alpha_0} \langle \xi, \partial_x \rangle + 2 \xi_{\alpha_0})(x_{\alpha_0} \langle \xi, \partial_x \rangle + \xi_{\alpha_0})(x_\alpha \langle \xi, \partial_x \rangle) F(x, \xi) = 0 \\ \Leftrightarrow \quad &\lim_{x_{\alpha_0} \to 0} \langle \xi, \partial_x \rangle^2 x_{\alpha_0}^3 \sum_{\substack{\alpha,\beta \in \mathcal{H} \\ \alpha \neq \beta}} u'_\alpha u_\beta \partial_{\xi,\alpha} \widetilde{P}_2^\beta = 0 \end{aligned} \tag{4.4}$$

for each $\alpha_0 \in \mathcal{H}$. Here, $\lim_{x_{\alpha_0}\to 0} G(x)$ is the limit $\lim_{x_{\alpha_0}\to 0} \hat{G}(x_{\alpha_0}, x_{\alpha_0^\perp})$, where $\hat{G}(x_{\alpha_0}, x_{\alpha_0^\perp})$ is the expression of $G(x)$ as a function of x_{α_0}, $x_{\alpha_0^\perp}$.

Theorem 4.2 *If $\widetilde{P}_0$ is not a polynomial in $\xi_1^2+\xi_2^2$, then, for each $\alpha_0 \in \mathcal{H}$,*

$$\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle \alpha_0, \beta\rangle}{\langle \alpha_0^\perp, \beta\rangle^3} C_\beta = 0 \tag{4.5}$$

holds.

PROOF. Since $u_\alpha(t) = C_\alpha/t^2$ and

$$\lim_{x_{\alpha_0}\to 0} u''_\beta = \lim_{x_{\alpha_0}\to 0} (-2)(-3)C_\beta \left(\frac{\langle \alpha_0, \beta\rangle x_{\alpha_0} + \langle \alpha_0^\perp, \beta\rangle x_{\alpha_0^\perp}}{|\alpha_0|^2} \right)^{-4} = \frac{6C_\beta |\alpha_0|^8}{\langle \alpha_0^\perp, \beta\rangle^4 x^4_{\alpha_0^\perp}},$$

we have

$$\begin{aligned}
&\lim_{x_{\alpha_0}\to 0} \langle \xi, \partial_x\rangle^2 x^3_{\alpha_0} \sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}} u'_\alpha u_\beta \partial_{\xi,\alpha} \widetilde{P}^\beta_2 = 0 \\
\Leftrightarrow\quad & \lim_{x_{\alpha_0}\to 0} \langle \xi, \partial_x\rangle^2 x^3_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} C_{\alpha_0} \left(-2x^{-3}_{\alpha_0} u_\beta \partial_{\xi,\alpha_0} \widetilde{P}^\beta_2 + x^{-2}_{\alpha_0} u'_\beta \partial_{\xi,\beta} \widetilde{P}^{\alpha_0}_2 \right) = 0 \\
\Leftrightarrow\quad & \lim_{x_{\alpha_0}\to 0} C_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \left(-2u''_\beta \xi^2_\beta \partial_{\xi,\alpha_0} \widetilde{P}^\beta_2 + 2u''_\beta \xi_{\alpha_0} \xi_\beta \partial_{\xi,\beta} \widetilde{P}^{\alpha_0}_2 \right) = 0 \\
\Leftrightarrow\quad & C_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle \alpha_0^\perp, \beta\rangle^4} \xi_\beta (\xi_\beta \partial_{\xi,\alpha_0} \widetilde{P}^\beta_2 - \xi_{\alpha_0} \partial_{\xi,\beta} \widetilde{P}^{\alpha_0}_2) = 0 \\
\Leftrightarrow\quad & C_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle \alpha_0, \beta\rangle C_\beta}{\langle \alpha_0^\perp, \beta\rangle^4} \xi_\beta (\widetilde{P}^{\alpha_0}_2 - \widetilde{P}^\beta_2) = 0 \\
\Leftrightarrow\quad & C_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle \alpha_0, \beta\rangle C_\beta}{\langle \alpha_0^\perp, \beta\rangle^4} \frac{\xi_\beta \partial_{\xi,\alpha_0} \widetilde{P}_0 - \xi_{\alpha_0} \partial_{\xi,\beta} \widetilde{P}_0}{\xi_{\alpha_0}} = 0 \\
\Leftrightarrow\quad & C_{\alpha_0} \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle \alpha_0, \beta\rangle C_\beta}{\langle \alpha_0^\perp, \beta\rangle^3} \frac{D_\theta \widetilde{P}_0}{\xi_{\alpha_0}} = 0.
\end{aligned}$$

Here, we used $\xi_\beta \partial_{\xi,\alpha} \widetilde{P}^\beta_2 = \xi_\beta \partial_{\xi,\alpha} (\partial_{\xi,\beta} \widetilde{P}_0/\xi_\beta) = \partial_{\xi,\alpha}\partial_{\xi,\beta} \widetilde{P}_0 - \langle \alpha, \beta\rangle \widetilde{P}^\beta_2$ and (3.1). □

Corollary 4.3 *Under the assumption of Theorem 4.2, if $\mathcal{H}$ consists of two vectors, then they cross at right angles and the singular locus is of type $A_1 \times A_1$.*

PROOF. Let $\mathcal{H} = \{\alpha, \beta\}$. Then by Theorem 4.2, we have $\langle \alpha, \beta\rangle C_\beta / \langle \alpha^\perp, \beta\rangle^3 = 0$. Since $C_\beta \neq 0$, this implies $\langle \alpha, \beta\rangle = 0$. □

Next, let us consider the last term in (4.2). Let $N := \#\mathcal{H}$ and

$$d_{\alpha,\beta,\gamma} := \frac{\langle \beta, \gamma\rangle}{\langle \beta^\perp, \gamma\rangle^3 C_\alpha} + \frac{\langle \gamma, \alpha\rangle}{\langle \gamma^\perp, \alpha\rangle^3 C_\beta} + \frac{\langle \alpha, \beta\rangle}{\langle \alpha^\perp, \beta\rangle^3 C_\gamma}.$$

Lemma 4.4 *The constant $d_{\alpha,\beta,\gamma}$ is skew-symmetric with respect to α, β, γ and satisfies*

$$\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha,\beta}} C_\gamma d_{\alpha,\beta,\gamma} = N\frac{\langle\alpha,\beta\rangle}{\langle\alpha^\perp,\beta\rangle^3}. \tag{4.6}$$

Proof. The first statement follows from $\langle\alpha^\perp,\beta\rangle = -\langle\beta^\perp,\alpha\rangle$.

The second statement is a consequence of (4.5):

$$\begin{aligned}\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha,\beta}} C_\gamma d_{\alpha,\beta,\gamma} &= \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha,\beta}} \left(\frac{1}{C_\alpha}\frac{\langle\beta,\gamma\rangle C_\gamma}{\langle\beta^\perp,\gamma\rangle^3} + \frac{1}{C_\beta}\frac{\langle\gamma,\alpha\rangle C_\gamma}{\langle\gamma^\perp,\alpha\rangle^3} + \frac{\langle\alpha,\beta\rangle}{\langle\alpha^\perp,\beta\rangle^3}\right)\\ &= -\frac{1}{C_\alpha}\frac{\langle\beta,\alpha\rangle C_\alpha}{\langle\beta^\perp,\alpha\rangle^3} - \frac{1}{C_\beta}\frac{\langle\beta,\alpha\rangle C_\beta}{\langle\beta^\perp,\alpha\rangle^3} + (N-2)\frac{\langle\alpha,\beta\rangle}{\langle\alpha^\perp,\beta\rangle^3}\\ &= N\frac{\langle\alpha,\beta\rangle}{\langle\alpha^\perp,\beta\rangle^3}.\end{aligned}$$

□

Since $u_\alpha(t) = C_\alpha/t^2$ and

$$\langle\beta^\perp,\gamma\rangle\alpha + \langle\gamma^\perp,\alpha\rangle\beta + \langle\alpha^\perp,\beta\rangle\gamma = 0 \tag{4.7}$$

for any α, β, $\gamma\in\mathcal{H}$, we have

$$\begin{vmatrix} \langle\beta^\perp,\gamma\rangle^3 C_\alpha & \langle\gamma^\perp,\alpha\rangle^3 C_\beta & \langle\alpha^\perp,\beta\rangle^3 C_\gamma \\ \langle\beta^\perp,\gamma\rangle u_\alpha & \langle\gamma^\perp,\alpha\rangle u_\beta & \langle\alpha^\perp,\beta\rangle u_\gamma \\ u'_\alpha & u'_\beta & u'_\gamma \end{vmatrix} = 0. \tag{4.8}$$

By Corollary 3.4, $D_\theta\widetilde{P}_0/\xi_\alpha\xi_\beta\xi_\gamma$ is a polynomial in ξ and it is symmetric with respect to α,β,γ. Then by (4.8), we have

$$\sum_{\substack{\alpha,\beta,\gamma\in\mathcal{H}\\ \alpha\neq\beta\neq\gamma\neq\alpha}} \langle\alpha^\perp,\beta\rangle^3\langle\gamma^\perp,\alpha\rangle C_\gamma d_{\alpha,\beta,\gamma}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma}u'_\alpha u_\beta = 0. \tag{4.9}$$

Lemma 4.5 *For $\alpha,\beta\in\mathcal{H}$, let*

$$\widetilde{P}_4^{\alpha,\beta} = -\frac{\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta}\widetilde{P}_2^\alpha + \langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}\widetilde{P}_2^\beta}{\langle\alpha^\perp,\beta\rangle\xi_\gamma} + \frac{\langle\alpha^\perp,\beta\rangle^3}{N}\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha,\beta,\gamma}}\langle\delta^\perp,\gamma\rangle C_\delta d_{\alpha,\beta,\delta}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta}, \tag{4.10}$$

where γ is any vector in $\mathcal{H}$ other than α,β. Then, $\widetilde{P}_4^{\alpha,\beta}$ is a polynomial in ξ and satisfies

$$\xi_\alpha\widetilde{P}_4^{\alpha,\beta} = \partial_{\xi,\alpha}\widetilde{P}_2^\beta + \frac{\langle\alpha^\perp,\beta\rangle^3}{N}\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha,\beta}}\langle\delta^\perp,\alpha\rangle C_\delta d_{\alpha,\beta,\delta}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\delta}. \tag{4.11}$$

PROOF. By Corollary 3.4, $D_\theta\widetilde{P}_0/\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta$ is a polynomial. Therefore, to prove $\widetilde{P}_4^{\alpha,\beta}$ being a polynomial, we have only to show that $\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta}\widetilde{P}_2^\alpha+\langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}\widetilde{P}_2^\beta$ is divisible by ξ_γ. Since

$$
\begin{aligned}
&\xi_\alpha\xi_\beta(\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta}\widetilde{P}_2^\alpha+\langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}\widetilde{P}_2^\beta)+\langle\alpha^\perp,\beta\rangle\xi_\gamma\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_0\\
&=\langle\gamma^\perp,\alpha\rangle\xi_\beta(\partial_{\xi,\beta}\partial_{\xi,\alpha}\widetilde{P}_0-\langle\alpha,\beta\rangle\widetilde{P}_2^\alpha)+\langle\beta^\perp,\gamma\rangle\xi_\alpha(\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_0-\langle\alpha,\beta\rangle\widetilde{P}_2^\beta)+\langle\alpha^\perp,\beta\rangle\xi_\gamma\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_0\\
&=(\langle\gamma^\perp,\alpha\rangle\xi_\beta+\langle\beta^\perp,\gamma\rangle\xi_\alpha+\langle\alpha^\perp,\beta\rangle\xi_\gamma)\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_0-\langle\alpha,\beta\rangle(\langle\gamma^\perp,\alpha\rangle\xi_\beta\widetilde{P}_2^\alpha+\langle\beta^\perp,\gamma\rangle\xi_\alpha\widetilde{P}_2^\beta)\\
&=\langle\alpha,\beta\rangle\{(\langle\alpha^\perp,\beta\rangle\xi_\gamma+\langle\beta^\perp,\gamma\rangle\xi_\alpha)\widetilde{P}_2^\alpha+(\langle\gamma^\perp,\alpha\rangle\xi_\beta+\langle\alpha^\perp,\beta\rangle\xi_\gamma)\widetilde{P}_2^\beta\}\\
&=\langle\alpha,\beta\rangle\langle\alpha^\perp,\beta\rangle\xi_\gamma(\widetilde{P}_2^\alpha+\widetilde{P}_2^\beta)+\langle\alpha,\beta\rangle(\langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}+\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta})\widetilde{P}_0\\
&=\langle\alpha,\beta\rangle\langle\alpha^\perp,\beta\rangle\xi_\gamma(\widetilde{P}_2^\alpha+\widetilde{P}_2^\beta-\widetilde{P}_2^\gamma),
\end{aligned}
$$

we have

$$
\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta}\widetilde{P}_2^\alpha+\langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}\widetilde{P}_2^\beta=\frac{\langle\alpha^\perp,\beta\rangle\xi_\gamma}{\xi_\alpha\xi_\beta}\{-\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_0+\langle\alpha,\beta\rangle(\widetilde{P}_2^\alpha+\widetilde{P}_2^\beta-\widetilde{P}_2^\gamma)\}.
$$

By the uniqueness of factorization, this is divisible by ξ_γ.

Let us prove (4.11). Firstly, we have

$$
\begin{aligned}
&-\xi_\alpha\frac{\langle\gamma^\perp,\alpha\rangle\partial_{\xi,\beta}\widetilde{P}_2^\alpha+\langle\beta^\perp,\gamma\rangle\partial_{\xi,\alpha}\widetilde{P}_2^\beta}{\langle\alpha^\perp,\beta\rangle\xi_\gamma}-\partial_{\xi,\alpha}\widetilde{P}_2^\beta\\
&\qquad=-\frac{\langle\gamma^\perp,\alpha\rangle\xi_\alpha\partial_{\xi,\beta}\widetilde{P}_2^\alpha+(\langle\beta^\perp,\gamma\rangle\xi_\alpha+\langle\alpha^\perp,\beta\rangle\xi_\gamma)\partial_{\xi,\alpha}\widetilde{P}_2^\beta}{\langle\alpha^\perp,\beta\rangle\xi_\gamma}\\
&\qquad=-\frac{\langle\gamma^\perp,\alpha\rangle(\xi_\alpha\partial_{\xi,\beta}\widetilde{P}_2^\alpha-\xi_\beta\partial_{\xi,\alpha}\widetilde{P}_2^\beta)}{\langle\alpha^\perp,\beta\rangle\xi_\gamma}\\
&\qquad=\langle\gamma^\perp,\alpha\rangle\langle\alpha,\beta\rangle\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma}.
\end{aligned}
$$

Here, we used (4.7) and calculated as in the proof of Theorem 4.2.

Secondly, since

$$
\frac{\langle\delta^\perp,\gamma\rangle}{\xi_\beta\xi_\gamma\xi_\delta}-\frac{\langle\delta^\perp,\alpha\rangle}{\xi_\alpha\xi_\beta\xi_\delta}=\frac{\langle\delta^\perp,\gamma\rangle\xi_\alpha+\langle\alpha^\perp,\delta\rangle\xi_\gamma}{\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta}=-\frac{\langle\gamma^\perp,\alpha\rangle}{\xi_\alpha\xi_\beta\xi_\gamma},
$$

we have

$$
\begin{aligned}
&\frac{\langle\alpha^\perp,\beta\rangle^3}{N}\left(\xi_\alpha\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha,\beta,\gamma}}\langle\delta^\perp,\gamma\rangle C_\delta d_{\alpha,\beta,\delta}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta}-\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha,\beta}}\langle\delta^\perp,\alpha\rangle C_\delta d_{\alpha,\beta,\delta}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\delta}\right)\\
&=\frac{\langle\alpha^\perp,\beta\rangle^3}{N}\left(-\frac{\langle\gamma^\perp,\alpha\rangle D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma}\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha,\beta,\gamma}}C_\delta d_{\alpha,\beta,\delta}-\langle\gamma^\perp,\alpha\rangle C_\gamma d_{\alpha,\beta,\gamma}\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma}\right)\\
&=\frac{\langle\alpha^\perp,\beta\rangle^3}{N}\left(-\frac{\langle\gamma^\perp,\alpha\rangle D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma}\right)N\frac{\langle\alpha,\beta\rangle}{\langle\alpha^\perp,\beta\rangle^3}\\
&=-\langle\gamma^\perp,\alpha\rangle\langle\alpha,\beta\rangle\frac{D_\theta\widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\gamma},
\end{aligned}
$$

by (4.6). Therefore, (4.11) is proved. □

Proposition 4.6 $\widetilde{P}_4^{\alpha,\beta}$ *satisfies*

$$\sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}} u'_\alpha u_\beta \partial_{\xi,\alpha}\widetilde{P}_2^\beta = \langle \xi, \partial_x\rangle \left(\sum_{\substack{\{\alpha,\beta\}\subset\mathcal{H}\\ \alpha\neq\beta}} u_\alpha u_\beta \widetilde{P}_4^{\alpha,\beta} \right). \tag{4.12}$$

Proof. By Lemma 4.5 and (4.9), we have

$$\begin{aligned}
&\langle \xi, \partial_x\rangle \left(\sum_{\substack{\{\alpha,\beta\}\subset\mathcal{H}\\ \alpha\neq\beta}} u_\alpha u_\beta \widetilde{P}_4^{\alpha,\beta} \right) - \sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}} u'_\alpha u_\beta \partial_{\xi,\alpha}\widetilde{P}_2^\beta \\
&= \sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}} u'_\alpha u_\beta \left(\xi_\alpha \widetilde{P}_4^{\alpha,\beta} - \partial_{\xi,\alpha}\widetilde{P}_2^\beta \right) \\
&= \sum_{\substack{\alpha,\beta,\delta\in\mathcal{H}\\ \alpha\neq\beta\neq\delta\neq\alpha}} u'_\alpha u_\beta \frac{\langle \alpha^\perp, \beta\rangle^3}{N} \langle \delta^\perp, \alpha\rangle C_\delta d_{\alpha,\beta,\delta} \frac{D_\theta \widetilde{P}_0}{\xi_\alpha\xi_\beta\xi_\delta} \\
&= 0.
\end{aligned}$$

□

Putting together these results, we obtain the explicit expression of $\widetilde{P}_4$.

Proposition 4.7 $\widetilde{P}_4$ *is expressed as*

$$\widetilde{P}_4 = \frac{1}{48}\sum_{\alpha\in\mathcal{H}} u''_\alpha \widetilde{P}_4^\alpha + \frac{1}{4}\sum_{\substack{\{\alpha,\beta\}\subset\mathcal{H}\\ \alpha\neq\beta}} u_\alpha u_\beta \widetilde{P}_4^{\alpha,\beta}.$$

Moreover, we can write down $\widetilde{P}_5$ explicitly by using Lemma 4.1.

5 Second condition for $\mathcal{H}$ and C_α

In the previous section, we obtained a condition (4.5) for $\mathcal{H}$ and C_α by investigating the pole of $\widetilde{P}_4$ at $x_{\alpha_0} = 0$. In this section, we investigate the pole of $\widetilde{P}_6$ at $x_{\alpha_0} = 0$ and obtain another condition for $\mathcal{H}$ and C_α.

Lemma 5.1 *Let*

$$\widetilde{P}_6^\alpha := \frac{1}{\xi_\alpha}\{24\partial_{\xi,\alpha}^5 \widetilde{P}_0 + (C_\alpha - 90|\alpha|^2)\partial_{\xi,\alpha}^3 \widetilde{P}_2^\alpha + C_\alpha \partial_{\xi,\alpha}\widetilde{P}_4^\alpha\}.$$

Then, $\widetilde{P}_6^\alpha$ *is a polynomial in* ξ *and* $\widetilde{P}_6$ *satisfies the following equation:*

$$\begin{aligned}
\langle \xi, \partial_x\rangle \widetilde{P}_6 &= \langle \xi, \partial_x\rangle \left(-\frac{1}{8}\langle \partial_\xi, \partial_x\rangle^2 \widetilde{P}_4 + \frac{1}{2}\langle \partial_\xi, \partial_x\rangle \widetilde{P}_5 - \frac{1}{5760}\sum_{\alpha\in\mathcal{H}} u_\alpha^{(4)} \widetilde{P}_6^\alpha \right) \\
&\quad + \frac{1}{96}\sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}} \{u_\alpha^{(3)} u_\beta (\partial_{\xi,\alpha}^3 \widetilde{P}_2^\beta - C_\alpha \partial_{\xi,\alpha}\widetilde{P}_4^{\alpha,\beta}) - u''_\alpha u'_\beta (3\partial_{\xi,\alpha}^2 \partial_{\xi,\beta}\widetilde{P}_2^\alpha + \partial_{\xi,\beta}\widetilde{P}_4^\alpha)\} \\
&\quad - \frac{1}{8}\sum_{\alpha\in\mathcal{H}} \sum_{\substack{\{\beta,\gamma\}\subset\mathcal{H}\\ \alpha\neq\beta\neq\gamma\neq\alpha}} u'_\alpha u_\beta u_\gamma \partial_{\xi,\alpha}\widetilde{P}_4^{\beta,\gamma}.
\end{aligned} \tag{5.1}$$

PROOF. Let us consider the expression $\widetilde{P}_0 = \sum_{k=0}^{m_0} c_k \xi_\alpha^k \xi_{\alpha^\perp}^{m_0-k}$. For this expression, we have

$$\widetilde{P}_6^\alpha = 15|\alpha|^6 c_5 (C_\alpha - 2|\alpha|^2)(C_\alpha - 6|\alpha|^2)\frac{\xi_{\alpha^\perp}^{m_0-5}}{\xi_\alpha} + (\text{a polynomial in } \xi).$$

Since c_5 satisfies $c_5(C_\alpha - 2|\alpha|^2)(C_\alpha - 6|\alpha|^2) = 0$ by Proposition 3.2, $\widetilde{P}_6^\alpha$ is a polynomial in ξ. The second assertion is a consequence of Lemma 3.1, (4.1), Proposition 4.6, $\Delta = [\langle\partial_x, \partial_\xi\rangle, \langle\xi, \partial_x\rangle]$ and $\langle\partial_x, \partial_\xi\rangle\Delta = (1/2)[\langle\partial_x, \partial_\xi\rangle^2, \langle\xi, \partial_x\rangle]$. □

We obtained (4.4) by investigating the pole of $\widetilde{P}_4$ at $x_{\alpha_0} = 0$. In the same way, we obtain

$$\begin{aligned}\lim_{x_{\alpha_0}\to 0}\langle\xi, \partial_x\rangle^4 x_{\alpha_0}^5 &\Bigg(\sum_{\substack{\alpha,\beta\in\mathcal{H}\\ \alpha\neq\beta}}\left\{u_\alpha^{(3)} u_\beta (\partial_{\xi,\alpha}^3 \widetilde{P}_2^\beta - C_\alpha \partial_{\xi,\alpha}\widetilde{P}_4^{\alpha,\beta}) - u_\alpha'' u_\beta' (3\partial_{\xi,\alpha}^2 \partial_{\xi,\beta}\widetilde{P}_2^\alpha + \partial_{\xi,\beta}\widetilde{P}_4^\alpha)\right\}\\ &\qquad -12\sum_{\alpha\in\mathcal{H}}\sum_{\substack{\{\beta,\gamma\}\subset\mathcal{H}\\ \alpha\neq\beta\neq\gamma\neq\alpha}} u_\alpha' u_\beta u_\gamma \partial_{\xi,\alpha}\widetilde{P}_4^{\beta,\gamma}\Bigg)\\ &= 0\end{aligned}$$

for each $\alpha_0 \in \mathcal{H}$. Since $u_\alpha = C_\alpha / x_\alpha^2$, this is equivalent to

$$S_1 + S_2 + S_3 = 0,$$

where

$$S_1 := -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp, \beta\rangle^6}\xi_\beta^3 \left\{\xi_\beta(\partial_{\xi,\alpha_0}^3 \widetilde{P}_2^\beta - C_{\alpha_0}\partial_{\xi,\alpha_0}\widetilde{P}_4^{\alpha_0,\beta}) + \xi_{\alpha_0}(3\partial_{\xi,\alpha_0}^2 \partial_{\xi,\beta}\widetilde{P}_2^{\alpha_0} + \partial_{\xi,\beta}\widetilde{P}_4^{\alpha_0})\right\},$$

$$S_2 := \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp, \beta\rangle^6}\xi_{\alpha_0}^2 \xi_\beta \left\{\xi_{\alpha_0}(\partial_{\xi,\beta}^3 \widetilde{P}_2^{\alpha_0} - C_\beta \partial_{\xi,\beta}\widetilde{P}_4^{\alpha_0,\beta}) + \xi_\beta(3\partial_{\xi,\beta}^2 \partial_{\xi,\alpha_0}\widetilde{P}_2^\beta + \partial_{\xi,\alpha_0}\widetilde{P}_4^\beta)\right\}$$

and

$$S_3 := \frac{1}{5}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{C_\beta C_\gamma}{\langle\alpha_0^\perp, \beta\rangle^4 \langle\alpha_0^\perp, \gamma\rangle^3}\xi_{\alpha_0}^2 (3\langle\alpha_0^\perp, \gamma\rangle\xi_\beta + 2\langle\alpha_0^\perp, \beta\rangle\xi_\gamma)(\xi_\beta \partial_{\xi,\alpha_0}\widetilde{P}_4^{\beta,\gamma} - \xi_{\alpha_0}\partial_{\xi,\beta}\widetilde{P}_4^{\alpha_0,\gamma}).$$

Theorem 5.2 *If $\widetilde{P}_0$ is not a polynomial in $\xi_1^2 + \xi_2^2$, then, for each $\alpha_0 \in \mathcal{H}$,*

$$(C_{\alpha_0} - 2|\alpha_0|^2)\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0, \beta\rangle|\beta|^2}{\langle\alpha_0^\perp, \beta\rangle^5} C_\beta = 0 \tag{5.2}$$

holds.

PROOF. The proof is divided into many lemmas. Let

$$\widetilde{Q}_{\alpha,\beta,\gamma} := D_\theta \widetilde{P}_0 / \xi_\alpha \xi_\beta \xi_\gamma \quad \text{and} \quad \widetilde{Q}_4^{\alpha,\beta} := \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha,\beta}} \langle\gamma^\perp, \alpha\rangle C_\gamma d_{\alpha,\beta,\gamma}\widetilde{Q}_{\alpha,\beta,\gamma}.$$

By (4.11), we have

$$
\begin{aligned}
&\xi_\beta \partial_{\xi,\alpha_0} \widetilde{P}_4^{\beta,\gamma} - \xi_{\alpha_0} \partial_{\xi,\beta} \widetilde{P}_4^{\alpha_0,\gamma} \\
&= \partial_{\xi,\alpha_0}(\xi_\beta \widetilde{P}_4^{\beta,\gamma}) - \partial_{\xi,\beta}(\xi_{\alpha_0} \widetilde{P}_4^{\alpha_0,\gamma}) - \langle \alpha_0, \beta\rangle (\widetilde{P}_4^{\beta,\gamma} - \widetilde{P}_4^{\alpha_0,\gamma}) \\
&= \frac{\langle \beta^\perp, \gamma\rangle^3}{N} \sum_{\substack{\delta\in\mathcal{H} \\ \delta\neq\beta,\gamma}} \langle \delta^\perp, \beta\rangle C_\delta d_{\beta,\gamma,\delta} \partial_{\xi,\alpha_0} \widetilde{Q}_{\beta,\gamma,\delta} - \frac{\langle \alpha_0^\perp, \gamma\rangle^3}{N} \partial_{\xi,\beta} \widetilde{Q}_4^{\alpha_0,\gamma} + \langle \alpha_0, \beta\rangle (\widetilde{P}_4^{\alpha_0,\gamma} - \widetilde{P}_4^{\beta,\gamma}).
\end{aligned}
$$

Therefore, we put

$$
S_3 = S_4 + S_5 + S_6 + S_7 + S_8,
$$

where

$$
\begin{aligned}
S_4 &:= \frac{3}{5} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle \alpha_0, \beta\rangle C_\beta C_\gamma}{\langle \alpha_0^\perp, \beta\rangle^4 \langle \alpha_0^\perp, \gamma\rangle^2} \xi_{\alpha_0}^2 \xi_\beta (\widetilde{P}_4^{\alpha_0,\gamma} - \widetilde{P}_4^{\beta,\gamma}), \\
S_5 &:= \frac{2}{5} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle \alpha_0, \beta\rangle C_\beta C_\gamma}{\langle \alpha_0^\perp, \beta\rangle^3 \langle \alpha_0^\perp, \gamma\rangle^3} \xi_{\alpha_0}^2 \xi_\gamma (\widetilde{P}_4^{\alpha_0,\gamma} - \widetilde{P}_4^{\beta,\gamma}) \\
&= \frac{2}{5} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle \alpha_0, \gamma\rangle C_\beta C_\gamma}{\langle \alpha_0^\perp, \beta\rangle^3 \langle \alpha_0^\perp, \gamma\rangle^3} \xi_{\alpha_0}^2 \xi_\beta (\widetilde{P}_4^{\alpha_0,\beta} - \widetilde{P}_4^{\beta,\gamma}), \\
S_6 &:= \frac{1}{5N} \sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H} \\ \alpha_0,\,\beta,\,\gamma,\,\delta \\ \text{are all different}}} \frac{\langle \beta^\perp, \gamma\rangle^3 \langle \delta^\perp, \beta\rangle C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle \alpha_0^\perp, \beta\rangle^4 \langle \alpha_0^\perp, \gamma\rangle^3} \xi_{\alpha_0}^2 (3\langle \alpha_0^\perp, \gamma\rangle \xi_\beta + 2\langle \alpha_0^\perp, \beta\rangle \xi_\gamma) \partial_{\xi,\alpha_0} \widetilde{Q}_{\beta,\gamma,\delta} \\
S_7 &:= \frac{1}{5N} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle \beta^\perp, \gamma\rangle^3 \langle \alpha_0^\perp, \beta\rangle C_\beta C_\gamma C_{\alpha_0} d_{\beta,\gamma,\alpha_0}}{\langle \alpha_0^\perp, \beta\rangle^4 \langle \alpha_0^\perp, \gamma\rangle^3} \xi_{\alpha_0}^2 (3\langle \alpha_0^\perp, \gamma\rangle \xi_\beta + 2\langle \alpha_0^\perp, \beta\rangle \xi_\gamma) \partial_{\xi,\alpha_0} \widetilde{Q}_{\beta,\gamma,\alpha_0} \\
&= \frac{1}{N} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle \beta^\perp, \gamma\rangle^3 C_{\alpha_0} C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle \alpha_0^\perp, \beta\rangle^3 \langle \alpha_0^\perp, \gamma\rangle^2} \xi_{\alpha_0}^2 \xi_\beta \partial_{\xi,\alpha_0} \widetilde{Q}_{\alpha_0,\beta,\gamma}, \\
S_8 &:= -\frac{1}{5N} \sum_{\substack{\beta,\gamma\in\mathcal{H} \\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{C_\beta C_\gamma}{\langle \alpha_0^\perp, \beta\rangle^4} \xi_{\alpha_0}^2 (3\langle \alpha_0^\perp, \gamma\rangle \xi_\beta + 2\langle \alpha_0^\perp, \beta\rangle \xi_\gamma) \partial_{\xi,\beta} \widetilde{Q}_4^{\alpha_0,\gamma}.
\end{aligned}
$$

Lemma 5.3 $\quad S_6 = 0$.

PROOF. For $\eta = (\eta_1, \eta_2)$, let

$$
\bar{S}_6(\eta) = \sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H} \\ \alpha_0,\,\beta,\,\gamma,\,\delta \\ \text{are all different}}} \frac{\langle \beta^\perp, \gamma\rangle^3 \langle \delta^\perp, \beta\rangle C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle \alpha_0^\perp, \beta\rangle^4 \langle \alpha_0^\perp, \gamma\rangle^3} \xi_{\alpha_0}^2 (3\langle \alpha_0^\perp, \gamma\rangle \eta_\beta + 2\langle \alpha_0^\perp, \beta\rangle \eta_\gamma) \partial_{\xi,\alpha_0} \widetilde{Q}_{\beta,\gamma,\delta}.
$$

We prove $\bar{S}_6(\eta) = 0$. If so, we have $S_6 = 0$, since $S_6 = \bar{S}_6(\xi)/5N$.

For an ordered triple $\{\beta, \gamma, \delta\} \subset \mathcal{H} \setminus \{\alpha_0\}$, let

$$
A_\beta := \langle \gamma^\perp, \delta\rangle \langle \alpha_0^\perp, \beta\rangle, \qquad A_\gamma := \langle \delta^\perp, \beta\rangle \langle \alpha_0^\perp, \gamma\rangle, \qquad A_\delta := \langle \beta^\perp, \gamma\rangle \langle \alpha_0^\perp, \delta\rangle.
$$

Note that they satisfy $\sigma(A_\varepsilon) = (\mathrm{sgn}\sigma) A_{\sigma(\varepsilon)}$ ($\varepsilon \in \{\beta, \gamma, \delta\}$) for a permutation σ of $\{\beta, \gamma, \delta\}$, and

$$
A_\beta + A_\gamma + A_\delta = 0
$$

by (4.7). Hereafter, we denote by $\sum_{\beta,\gamma,\delta}{}' F(b,c,d)$ the sum $F(\beta,\gamma,\delta)+F(\gamma,\delta,\beta)+F(\delta,\beta,\gamma)$.

Firstly, we have

$$\begin{aligned}
\bar{S}_6(\eta) = &\sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\delta^\perp,\beta\rangle\langle\alpha_0^\perp,\gamma\rangle\langle\alpha_0^\perp,\delta\rangle^4 C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^4\langle\alpha_0^\perp,\delta\rangle^4}\xi_{\alpha_0}^2\\
&\qquad\times(3\langle\alpha_0^\perp,\gamma\rangle\eta_\beta+2\langle\alpha_0^\perp,\beta\rangle\eta_\gamma)\partial_{\xi,\alpha_0}\widetilde{Q}_{\beta,\gamma,\delta}\\
= &\sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{\langle\alpha_0^\perp,\delta\rangle A_\gamma A_\delta^3\xi_{\alpha_0}^2 C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^4\langle\alpha_0^\perp,\delta\rangle^4}(3\langle\alpha_0^\perp,\gamma\rangle\eta_\beta+2\langle\alpha_0^\perp,\beta\rangle\eta_\gamma)\partial_{\xi,\alpha_0}\widetilde{Q}_{\beta,\gamma,\delta}\\
= &\sum_{\substack{\{\beta,\,\gamma,\,\delta\}\,\subset\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^4\langle\alpha_0^\perp,\delta\rangle^4}(3S_9+2S_{10})\xi_{\alpha_0}^2\partial_{\xi,\alpha_0}\widetilde{Q}_{\beta,\gamma,\delta},
\end{aligned}$$

where

$$\begin{aligned}
S_9 &:= \sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,d\rangle A_d^3(\langle\alpha_0^\perp,c\rangle A_c\eta_b-\langle\alpha_0^\perp,b\rangle A_b\eta_c),\\
S_{10} &:= \sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,d\rangle A_d^3(\langle\alpha_0^\perp,b\rangle A_c\eta_c-\langle\alpha_0^\perp,c\rangle A_b\eta_b).
\end{aligned}$$

Since

$$\begin{aligned}
S_9 =&\sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,b\rangle\langle\alpha_0^\perp,c\rangle A_bA_c\eta_d(A_c^2-A_b^2) && = A_\beta A_\gamma A_\delta\sum_{\beta,\gamma,\delta}{}'(A_b-A_c)\langle\alpha_0^\perp,b\rangle\langle\alpha_0^\perp,c\rangle\eta_d\\
=&A_\beta A_\gamma A_\delta\sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,b\rangle A_b(\langle\alpha_0^\perp,c\rangle\eta_d-\langle\alpha_0^\perp,d\rangle\eta_c) && = -A_\beta A_\gamma A_\delta\eta_{\alpha_0}\sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,b\rangle\langle c^\perp,d\rangle A_b\\
=&-A_\beta A_\gamma A_\delta(A_\beta^2+A_\gamma^2+A_\delta^2)\eta_{\alpha_0} && = 2A_\beta A_\gamma A_\delta(A_\beta A_\gamma+A_\gamma A_\delta+A_\delta A_\beta)\eta_{\alpha_0}
\end{aligned}$$

and

$$\begin{aligned}
S_{10} =&\sum_{\beta,\gamma,\delta}{}'\langle\alpha_0^\perp,d\rangle A_d^3\{\langle\alpha_0^\perp,b\rangle(A_c-A_b)\eta_c+A_b(\langle\alpha_0^\perp,b\rangle\eta_c-\langle\alpha_0^\perp,c\rangle\eta_b)\}\\
=&\sum_{\beta,\gamma,\delta}{}'\{\langle\alpha_0^\perp,d\rangle\langle\alpha_0^\perp,b\rangle A_d^2(A_b^2-A_c^2)\eta_c-\langle\alpha_0^\perp,d\rangle\langle b^\perp,c\rangle A_d^3A_b\eta_{\alpha_0}\}\\
=&\sum_{\beta,\gamma,\delta}{}'\{A_b^2A_c^2\langle\alpha_0^\perp,c\rangle(\langle\alpha_0^\perp,b\rangle\eta_d-\langle\alpha_0^\perp,d\rangle\eta_b)-A_d^4A_b\eta_{\alpha_0}\}\\
=&\eta_{\alpha_0}\sum_{\beta,\gamma,\delta}{}'(A_b^2A_c^3-A_b^4A_c)\\
=&\eta_{\alpha_0}\sum_{\beta,\gamma,\delta}{}'A_b^2A_c(A_c^2-A_b^2)\\
=&\eta_{\alpha_0}\sum_{\beta,\gamma,\delta}{}'A_b^2A_cA_d(A_b-A_c)\\
=&A_\beta A_\gamma A_\delta\eta_{\alpha_0}\sum_{\beta,\gamma,\delta}{}'(A_b^2-A_bA_c)\\
=&-3A_\beta A_\gamma A_\delta(A_\beta A_\gamma+A_\gamma A_\delta+A_\delta A_\beta)\eta_{\alpha_0},
\end{aligned}$$

we have $3S_9 + 2S_{10} = 0$ and $\bar{S}_6(\eta) = 0$. $\square$

Lemma 5.4 $S_4 = -\frac{3}{5}(S_{11} + S_{12})$, *where*

$$S_{11} := \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^6}\xi_{\alpha_0}^2(\xi_\beta \widetilde{P}_4^{\alpha_0,\beta} - \partial_{\xi,\beta}\widetilde{P}_2^\beta), \quad S_{12} := \frac{1}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\alpha_0,\beta\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}^2\widetilde{Q}_4^{\beta,\gamma}.$$

Proof. We use (3.1), (4.5) and (4.11). Firstly,

$$\begin{aligned}\frac{5}{3}S_4 = &\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}\\
&\times\left\{\xi_\beta\left(\partial_{\xi,\alpha_0}\widetilde{P}_2^\gamma + \frac{\langle\alpha_0^\perp,\gamma\rangle^3}{N}\widetilde{Q}_4^{\alpha_0,\gamma}\right) - \xi_{\alpha_0}\left(\partial_{\xi,\beta}\widetilde{P}_2^\gamma + \frac{\langle\beta^\perp,\gamma\rangle^3}{N}\widetilde{Q}_4^{\beta,\gamma}\right)\right\}\\
= &\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}D_\theta\widetilde{P}_2^\gamma + \frac{1}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle\langle\alpha_0^\perp,\gamma\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}\xi_\beta\widetilde{Q}_4^{\alpha_0,\gamma} - S_{12}.\end{aligned}$$

The lemma is a consequence of the following calculations:

(i)
$$\begin{aligned}&\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}D_\theta\widetilde{P}_2^\gamma\\
&= \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}} \frac{C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}D_\theta\widetilde{P}_2^\gamma \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0,\gamma}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3} = -\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\not\alpha_0}} \frac{\langle\alpha_0,\gamma\rangle C_\gamma^2}{\langle\alpha_0^\perp,\gamma\rangle^5}\xi_{\alpha_0}D_\theta\widetilde{P}_2^\gamma\\
&= -\sum_{\substack{\gamma\in\mathcal{H}\\ \beta\not\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^5}\xi_{\alpha_0}D_\theta\widetilde{P}_2^\beta.\end{aligned}$$

(ii)
$$\begin{aligned}&\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle\langle\alpha_0^\perp,\gamma\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}\xi_\beta\widetilde{Q}_4^{\alpha_0,\gamma}\\
&= \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}\xi_\beta \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}}\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\alpha_0,\gamma}} \langle\alpha_0^\perp,\gamma\rangle\langle\delta^\perp,\alpha_0\rangle C_\gamma C_\delta d_{\alpha_0,\gamma,\delta}\widetilde{Q}_{\alpha_0,\gamma,\delta}\\
&= \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}\xi_\beta \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}} \langle\alpha_0^\perp,\gamma\rangle\langle\beta^\perp,\alpha_0\rangle C_\gamma C_\beta d_{\alpha_0,\gamma,\beta}\widetilde{Q}_{\alpha_0,\gamma,\beta}\\
&= -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}\xi_\beta \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}} \langle\gamma^\perp,\alpha_0\rangle C_\gamma d_{\alpha_0,\beta,\gamma}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
&= -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}\xi_\beta\widetilde{Q}_4^{\alpha_0,\beta}.\end{aligned}$$

(iii)
$$\begin{aligned}&\langle\alpha_0^\perp,\beta\rangle\xi_{\alpha_0}D_\theta\widetilde{P}_2^\beta + \frac{\langle\alpha_0^\perp,\beta\rangle^3}{N}\xi_{\alpha_0}\xi_\beta\widetilde{Q}_4^{\alpha_0,\beta}\\
&= \xi_{\alpha_0}\xi_\beta\left(\partial_{\alpha_0}\widetilde{P}_2^\beta + \frac{\langle\alpha_0^\perp,\beta\rangle^3}{N}\widetilde{Q}_4^{\alpha_0,\beta}\right) - \xi_{\alpha_0}^2\partial_\beta\widetilde{P}_2^\beta = \xi_{\alpha_0}^2(\xi_\beta\widetilde{P}_4^{\alpha_0,\beta} - \partial_\beta\widetilde{P}_2^\beta).\end{aligned}$$

$\square$

Let us rewrite S_5 analogously. By (4.5), there exists a constant K_{α_0} such that

$$\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\beta} = K_{\alpha_0}\partial_{\xi,\alpha_0^\perp}.$$

Lemma 5.5 $S_5 = -\frac{2}{5}(S_{11}+S_{12}) - S_{13} + S_{14}$, *where*

$$S_{13} = \frac{2K_{\alpha_0}}{5}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\partial_{\xi,\alpha_0^\perp}\widetilde{P}_2^\beta,$$

$$S_{14} = \frac{1}{N}\sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\delta^\perp,\beta\rangle\langle\alpha_0,\beta\rangle C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}^2\widetilde{Q}_{\beta,\gamma,\delta}.$$

Proof. Let us calculate $S_5 + 2S_{12}/5 - S_{14}$:

$$\begin{aligned}
&S_5 + \frac{2}{5}S_{12} - S_{14}\\
&\quad= \frac{2}{5}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\beta\widetilde{P}_4^{\alpha_0,\beta}\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}}\frac{\langle\alpha_0,\gamma\rangle C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^3}\\
&\qquad - \frac{2}{5}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\gamma\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^3}\xi_{\alpha_0}^2\left(\partial_{\xi,\beta}\widetilde{P}_2^\gamma + \frac{\langle\beta^\perp,\gamma\rangle^3}{N}\widetilde{Q}_4^{\beta,\gamma}\right) + \frac{2}{5}S_{12} - S_{14}\\
&\quad= -\frac{2}{5}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^6}\xi_{\alpha_0}^2\xi_\beta\widetilde{P}_4^{\alpha_0,\beta} - \frac{2}{5}\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}}\frac{\langle\alpha_0,\gamma\rangle C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^3}\xi_{\alpha_0}^2\left(\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0,\gamma}}\frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\beta}\right)\widetilde{P}_2^\gamma\\
&\qquad - \frac{2}{5N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\alpha_0,\gamma\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^3}\xi_{\alpha_0}^2\widetilde{Q}_4^{\beta,\gamma} + \frac{2}{5}S_{12} - S_{14}\\
&\quad= -\frac{2}{5}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^6}\xi_{\alpha_0}^2\xi_\beta\widetilde{P}_4^{\alpha_0,\beta} - \frac{2}{5}\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}}\frac{\langle\alpha_0,\gamma\rangle C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^3}\xi_{\alpha_0}^2\left(K_{\alpha_0}\partial_{\xi,\alpha_0^\perp} - \frac{C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^3}\partial_{\xi,\gamma}\right)\widetilde{P}_2^\gamma \qquad (5.3)\\
&\qquad - \frac{1}{5N}\sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\delta^\perp,\beta\rangle C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^3}\xi_{\alpha_0}^2\\
&\qquad\qquad\qquad \times(3\langle\alpha_0^\perp,\gamma\rangle\langle\alpha_0,\beta\rangle + 2\langle\alpha_0^\perp,\beta\rangle\langle\alpha_0,\gamma\rangle)\widetilde{Q}_{\beta,\gamma,\delta} \qquad (5.4)\\
&\qquad - \frac{2}{5N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\beta^\perp,\gamma\rangle^3 C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^3}(\langle\alpha_0^\perp,\gamma\rangle\langle\alpha_0,\beta\rangle - \langle\alpha_0^\perp,\beta\rangle\langle\alpha_0,\gamma\rangle)\xi_{\alpha_0}^2\widetilde{Q}_{\alpha_0,\beta,\gamma}. \qquad (5.5)
\end{aligned}$$

The terms in (5.3) are equal to $-2S_{11}/5 - S_{13}$. The remaining terms vanish, since (5.4) is $-\bar{S}_6(\alpha_0)/5N$ and the summand in (5.5) is skew-symmetric with respect to β,γ. $\square$

Lemma 5.6 *$S_8 = S_{13} + S_{15}$, where* $S_{15} := \dfrac{1}{N} \displaystyle\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\beta}\widetilde{Q}_4^{\alpha_0,\beta}.$

PROOF. This lemma is a consequence of the following calculations:

(i)
$$\begin{aligned}
&\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0^\perp,\gamma\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\beta}\widetilde{Q}_4^{\alpha_0,\gamma}\\
&= \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}^2\xi_\beta \sum_{\substack{\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \langle\alpha_0^\perp,\gamma\rangle\langle\delta^\perp,\alpha_0\rangle C_\gamma C_\delta d_{\alpha_0,\gamma,\delta}\partial_{\xi,\beta}\widetilde{Q}_{\alpha_0,\gamma,\delta}\\
&\qquad + \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^4}\xi_{\alpha_0}^2\xi_\beta \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}} \langle\alpha_0^\perp,\gamma\rangle\langle\beta^\perp,\alpha_0\rangle C_\gamma C_\beta d_{\alpha_0,\gamma,\beta}\partial_{\xi,\beta}\widetilde{Q}_{\alpha_0,\gamma,\beta}\\
&= -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\beta}\left(\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}} \langle\gamma^\perp,\alpha_0\rangle C_\gamma d_{\alpha_0,\beta,\gamma}\widetilde{Q}_{\alpha_0,\beta,\gamma}\right)\\
&= -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\beta}\widetilde{Q}_4^{\alpha_0,\beta}\\
&= -NS_{15}.
\end{aligned}$$

(ii)
$$\begin{aligned}
&\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\gamma\partial_{\xi,\beta}\widetilde{Q}_4^{\alpha_0,\gamma}\\
&= \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}} C_\gamma\xi_{\alpha_0}^2\xi_\gamma\left(\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0,\gamma}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\beta}\right)\widetilde{Q}_4^{\alpha_0,\gamma}\\
&= \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}} C_\gamma\xi_{\alpha_0}^2\xi_\gamma\left(K_{\alpha_0}\partial_{\xi,\alpha_0^\perp} - \frac{C_\gamma}{\langle\alpha_0^\perp,\gamma\rangle^3}\partial_{\xi,\gamma}\right)\widetilde{Q}_4^{\alpha_0,\gamma}\\
&= K_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} C_\beta\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\alpha_0^\perp}\widetilde{Q}_4^{\alpha_0,\beta} - NS_{15}.
\end{aligned}$$

(iii)
$$\begin{aligned}
&\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} C_\beta\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\alpha_0^\perp}\widetilde{Q}_4^{\alpha_0,\beta}\\
&= \xi_{\alpha_0}^2\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} C_\beta\xi_\beta\partial_{\xi,\alpha_0^\perp}\left(\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}} \langle\gamma^\perp,\alpha_0\rangle C_\gamma d_{\alpha_0,\beta,\gamma}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}\xi_\beta\xi_\gamma}\right)\\
&= \xi_{\alpha_0}^2\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \langle\gamma^\perp,\alpha_0\rangle C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}\left\{\partial_{\xi,\alpha_0^\perp}\left(\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}\xi_\gamma}\right) - \langle\alpha_0^\perp,\beta\rangle\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}\xi_\beta\xi_\gamma}\right\}\\
&= \xi_{\alpha_0}^2\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}} \langle\gamma^\perp,\alpha_0\rangle C_\gamma\partial_{\xi,\alpha_0^\perp}\left(\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}\xi_\gamma}\right)\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0,\gamma}} C_\beta d_{\alpha_0,\beta,\gamma}
\end{aligned}$$

$$
\begin{aligned}
&= -\xi_{\alpha_0}^2 \sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0}} \langle\gamma^\perp,\alpha_0\rangle C_\gamma \partial_{\xi,\alpha_0^\perp}\left(\frac{D_\theta \widetilde{P}_0}{\xi_{\alpha_0}\xi_\gamma}\right)\times N\frac{\langle\alpha_0,\gamma\rangle}{\langle\alpha_0^\perp,\gamma\rangle^3}\\
&= -N\xi_{\alpha_0}^2 \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\alpha_0^\perp}(\widetilde{P}_2^\beta-\widetilde{P}_2^{\alpha_0})\\
&= -N\xi_{\alpha_0}^2 \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\alpha_0^\perp}\widetilde{P}_2^\beta + N\xi_{\alpha_0}^2\partial_{\xi,\alpha_0^\perp}\widetilde{P}_2^{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\\
&= -N\xi_{\alpha_0}^2 \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\partial_{\xi,\alpha_0^\perp}\widetilde{P}_2^\beta\\
&= -\frac{5NS_{13}}{2K_{\alpha_0}}.
\end{aligned}
$$

□

Lemma 5.7 $-S_{12}+S_{14}=S_{16}+S_{17}$, *where*

$$
\begin{aligned}
S_{16} :=& \frac{1}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3}(\langle\gamma^\perp,\alpha_0\rangle\langle\alpha_0,\beta\rangle-\langle\alpha_0^\perp,\beta\rangle\langle\alpha_0,\gamma\rangle)\xi_\beta^3\frac{\widetilde{Q}_{\alpha_0,\beta,\gamma}}{\xi_{\alpha_0}},\\
S_{17} :=& \frac{3}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\beta^\perp,\gamma\rangle\langle\alpha_0,\beta\rangle C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2\langle\alpha_0^\perp,\gamma\rangle}\xi_\beta\xi_\gamma\widetilde{Q}_{\alpha_0,\beta,\gamma}.
\end{aligned}
$$

PROOF.

$$
\begin{aligned}
N(-S_{12}+S_{14}) =& -\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\alpha_0,\beta\rangle C_\beta C_\gamma}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}^2\sum_{\substack{\delta\in\mathcal{H}\\ \delta\neq\beta,\gamma}}\langle\delta^\perp,\beta\rangle C_\delta d_{\beta,\gamma,\delta}\widetilde{Q}_{\beta,\gamma,\delta}\\
&+\sum_{\substack{\beta,\,\gamma,\,\delta\,\in\,\mathcal{H}\\ \alpha_0,\,\beta,\,\gamma,\,\delta\\ \text{are all different}}} \frac{\langle\beta^\perp,\gamma\rangle^3\langle\delta^\perp,\beta\rangle\langle\alpha_0,\beta\rangle C_\beta C_\gamma C_\delta d_{\beta,\gamma,\delta}}{\langle\alpha_0^\perp,\beta\rangle^4\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}^2\widetilde{Q}_{\beta,\gamma,\delta}\\
=& -\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}} \frac{\langle\alpha_0,\beta\rangle C_{\alpha_0}C_\beta C_\gamma d_{\beta,\gamma,\alpha_0}}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^2}(\langle\beta^\perp,\gamma\rangle\xi_{\alpha_0})^3\frac{\widetilde{Q}_{\beta,\gamma,\alpha_0}}{\xi_\alpha}\\
=& \sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\langle\alpha_0,\beta\rangle C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}\frac{\widetilde{Q}_{\alpha_0,\beta,\gamma}}{\xi_\alpha}\\
&\times\left(\frac{\langle\gamma^\perp,\alpha_0\rangle\xi_\beta^3}{\langle\alpha_0^\perp,\beta\rangle^3}+3\frac{\langle\beta^\perp,\gamma\rangle\xi_{\alpha_0}\xi_\beta\xi_\gamma^2}{\langle\alpha_0^\perp,\beta\rangle^2\langle\gamma^\perp,\alpha_0\rangle}+\frac{\xi_\gamma^3}{\langle\gamma^\perp,\alpha_0\rangle^2}\right)\\
=&N(S_{16}+S_{17}).
\end{aligned}
$$

□

We summarise the above calculations once.

Corollary 5.8 $S_3 = S_7 - S_{11} + S_{15} + S_{16} + S_{17}$.

Next, let us calculate the terms S_1, S_2.

Lemma 5.9 $S_1 + S_2 = S_{11} - S_{15} + S_{18} + S_{19} + S_{20}$, *where*

$$S_{18} = -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{1}{\xi_{\alpha_0}}(\xi_\beta^3\partial_{\xi,\alpha_0}^3\widetilde{P}_0 - 3\xi_{\alpha_0}\xi_\beta^3\partial_{\xi,\alpha_0}^2\widetilde{P}_2^\beta + 3\xi_{\alpha_0}^3\xi_\beta\xi_\beta\partial_{\xi,\beta}^2\widetilde{P}_2^{\alpha_0} - \xi_{\alpha_0}^3\partial_{\xi,\beta}^3\widetilde{P}_0),$$

$$S_{19} = -C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{\xi_\beta^3}{\xi_{\alpha_0}}(\xi_{\alpha_0}\widetilde{P}_4^{\alpha_0,\beta} - \partial_{\xi,\alpha_0}\widetilde{P}_2^{\alpha_0}),$$

$$S_{20} = -\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_\beta^3\partial_{\xi,\alpha_0}\widetilde{Q}_4^{\beta,\alpha_0}.$$

Proof. Since

$$\begin{aligned}
&\xi_\beta(\partial_{\xi,\alpha}^3\widetilde{P}_2^\beta - C_\alpha\partial_{\xi,\alpha}\widetilde{P}_4^{\alpha,\beta}) + \xi_\alpha(3\partial_{\xi,\alpha}^2\partial_{\xi,\beta}\widetilde{P}_2^\alpha + \partial_{\xi,\beta}\widetilde{P}_4^\alpha)\\
&\quad= \langle\alpha,\beta\rangle(-3\partial_{\xi,\alpha}^2\widetilde{P}_2^\beta + C_\alpha\widetilde{P}_4^{\alpha,\beta} - 3\partial_{\xi,\alpha}^2\widetilde{P}_2^\alpha - \widetilde{P}_4^\alpha) - 6|\alpha|^2\partial_{\xi,\alpha}\partial_{\xi,\beta}\widetilde{P}_2^\alpha + \partial_{\xi,\alpha}^3\partial_{\xi,\beta}\widetilde{P}_0\\
&\qquad\quad - C_\alpha\partial_{\xi,\alpha}\left(\partial_{\xi,\beta}\widetilde{P}_2^\alpha + \frac{\langle\beta^\perp,\alpha\rangle^3}{N}\widetilde{Q}_4^{\beta,\alpha}\right) + 3\partial_{\xi,\alpha}^3\partial_{\xi,\beta}\widetilde{P}_0 + \partial_{\xi,\beta}\{(C_\alpha + 6|\alpha|^2)\partial_{\xi,\alpha}\widetilde{P}_2^\alpha - 4\partial_{\xi,\alpha}^3\widetilde{P}_0\}\\
&\quad= \frac{\langle\alpha,\beta\rangle}{\xi_\alpha}\{-3\xi_\alpha\partial_{\xi,\alpha}^2\widetilde{P}_2^\beta - 3\xi_\alpha\partial_{\xi,\alpha}^2\widetilde{P}_2^\alpha - 6|\alpha|^2\partial_{\xi,\alpha}\widetilde{P}_2^\alpha + 4\partial_{\xi,\alpha}^3\widetilde{P}_0 + C_\alpha(\xi_\alpha\widetilde{P}_4^{\alpha,\beta} - \partial_{\xi,\alpha}\widetilde{P}_2^\alpha)\}\\
&\qquad\quad + \frac{1}{N}C_\alpha\langle\alpha^\perp,\beta\rangle^3\partial_{\xi,\alpha}\widetilde{Q}_4^{\beta,\alpha}\\
&\quad= \frac{\langle\alpha,\beta\rangle}{\xi_\alpha}\left\{-3\xi_\alpha\partial_{\xi,\alpha}^2\widetilde{P}_2^\beta + \partial_{\xi,\alpha}^3\widetilde{P}_0 + C_\alpha(\xi_\alpha\widetilde{P}_4^{\alpha,\beta} - \partial_{\xi,\alpha}\widetilde{P}_2^\alpha)\right\} + \frac{1}{N}C_\alpha\langle\alpha^\perp,\beta\rangle^3\partial_{\xi,\alpha}\widetilde{Q}_4^{\beta,\alpha},
\end{aligned}$$

we have

$$\begin{aligned}
&S_1 + S_2\\
&= -\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{1}{\xi_{\alpha_0}}(\xi_\beta^3\partial_{\xi,\alpha_0}^3\widetilde{P}_0 - 3\xi_{\alpha_0}\xi_\beta^3\partial_{\xi,\alpha_0}^2\widetilde{P}_2^\beta + 3\xi_{\alpha_0}^3\xi_\beta\xi_\beta\partial_{\xi,\beta}^2\widetilde{P}_2^{\alpha_0} - \xi_{\alpha_0}^3\partial_{\xi,\beta}^3\widetilde{P}_0)\\
&\quad - C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{\xi_\beta^3}{\xi_{\alpha_0}}(\xi_{\alpha_0}\widetilde{P}_4^{\alpha_0,\beta} - \partial_{\xi,\alpha_0}\widetilde{P}_2^{\alpha_0}) + \sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{\langle\alpha_0,\beta\rangle C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^6}\xi_{\alpha_0}^2(\xi_\beta\widetilde{P}_4^{\alpha_0,\beta} - \partial_{\xi,\beta}\widetilde{P}_2^\beta)\\
&\quad - \frac{C_{\alpha_0}}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_\beta^3\partial_{\xi,\alpha_0}\widetilde{Q}_4^{\beta,\alpha_0} - \frac{1}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}} \frac{C_\beta^2}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\beta}\widetilde{Q}_4^{\alpha_0,\beta}\\
&= S_{18} + S_{19} + S_{11} + S_{20} - S_{15}.
\end{aligned}$$

□

Lemma 5.10 $S_7 + S_{16} + S_{19} + S_{20} = 0$.

Proof.

$$
\begin{aligned}
&S_7+S_{20}\\
=&\frac{1}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{\langle\beta^\perp,\gamma\rangle^3 C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3\langle\alpha_0^\perp,\gamma\rangle^2}\xi_{\alpha_0}^2\xi_\beta\partial_{\xi,\alpha_0}\widetilde{Q}_{\alpha_0,\beta,\gamma}-\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\xi_\beta^3\partial_{\xi,\alpha_0}\widetilde{Q}_4^{\beta,\alpha_0}\\
=&\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3}\frac{\xi_\beta}{\xi_{\alpha_0}}\left(\frac{(\langle\beta^\perp,\gamma\rangle\xi_{\alpha_0})^3}{\langle\alpha_0^\perp,\gamma\rangle^2}-\langle\beta^\perp,\gamma\rangle\xi_{\alpha_0}\xi_\beta^2\right)\partial_{\xi,\alpha_0}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3}\frac{\xi_\beta}{\xi_{\alpha_0}}\partial_{\xi,\alpha_0}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
&\quad\times\left(\langle\alpha_0^\perp,\gamma\rangle\xi_\beta^3-3\langle\alpha_0^\perp,\beta\rangle\xi_\beta^2\xi_\gamma+3\frac{\langle\alpha_0^\perp,\beta\rangle^2}{\langle\alpha_0^\perp,\gamma\rangle}\xi_\beta\xi_\gamma^2-\frac{\langle\alpha_0^\perp,\beta\rangle^3}{\langle\alpha_0^\perp,\gamma\rangle^2}\xi_\gamma^3-\langle\alpha_0^\perp,\gamma\rangle\xi_\beta^3+\langle\alpha_0^\perp,\beta\rangle\xi_\beta^2\xi_\gamma\right)\\
=&\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}\frac{\partial_{\xi,\alpha_0}\widetilde{Q}_{\alpha_0,\beta,\gamma}}{\xi_{\alpha_0}}\left(-\frac{2\xi_\beta^3\xi_\gamma}{\langle\alpha_0^\perp,\beta\rangle^2}+\frac{3\xi_\beta^2\xi_\gamma^2}{\langle\alpha_0^\perp,\beta\rangle\langle\alpha_0^\perp,\gamma\rangle}-\frac{\xi_\beta\xi_\gamma^3}{\langle\alpha_0^\perp,\gamma\rangle^2}\right)\\
=&-\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2}\frac{\xi_\beta^3\xi_\gamma}{\xi_{\alpha_0}}\partial_{\xi,\alpha_0}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&-\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2}\frac{\xi_\beta^3}{\xi_{\alpha_0}}\left\{\partial_{\xi,\alpha_0}\left(\xi_\gamma\frac{\widetilde{P}_2^{\alpha_0}-\widetilde{P}_2^{\beta}}{\langle\alpha_0^\perp,\beta\rangle\xi_\gamma}\right)-\langle\gamma,\alpha_0\rangle\widetilde{Q}_{\alpha_0,\beta,\gamma}\right\}\\
=&-\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\left(\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}}C_\gamma d_{\alpha_0,\beta,\gamma}\right)\frac{\xi_\beta^3}{\xi_{\alpha_0}}\partial_{\xi,\alpha_0}(\widetilde{P}_2^{\alpha_0}-\widetilde{P}_2^{\beta})\\
&\quad+\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{\langle\gamma,\alpha_0\rangle C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2}\frac{\xi_\beta^3}{\xi_{\alpha_0}}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&-C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{\xi_\beta^3}{\xi_{\alpha_0}}\partial_{\xi,\alpha_0}(\widetilde{P}_2^{\alpha_0}-\widetilde{P}_2^{\beta})+\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{\langle\gamma,\alpha_0\rangle C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2}\frac{\xi_\beta^3}{\xi_{\alpha_0}}\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^6}\frac{\xi_\beta^3}{\xi_{\alpha_0}}\left(\partial_{\xi,\alpha_0}\widetilde{P}_2^{\beta}+\frac{\langle\alpha_0^\perp,\beta\rangle^3}{N}\widetilde{Q}_4^{\alpha_0,\beta}-\partial_{\xi,\alpha_0}\widetilde{P}_2^{\alpha_0}\right)\\
&\quad-\frac{C_{\alpha_0}}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^3}\frac{\xi_\beta^3}{\xi_{\alpha_0}}(\langle\alpha_0,\beta\rangle\langle\gamma^\perp,\alpha_0\rangle-\langle\gamma,\alpha_0\rangle\langle\alpha_0^\perp,\beta\rangle)\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&-S_{19}-S_{16}.
\end{aligned}
$$

□

Lemma 5.11 $S_{17}=3C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle|\beta|^2C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}.$

Proof.

$$
\begin{aligned}
S_{17} =&\frac{3}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle^2\langle\alpha_0^\perp,\gamma\rangle}(-\langle\gamma^\perp,\alpha_0\rangle|\beta|^2-\langle\alpha_0^\perp,\beta\rangle\langle\beta,\gamma\rangle)\xi_\beta\xi_\gamma\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&\frac{3}{N}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{|\beta|^2C_{\alpha_0}C_\beta}{\langle\alpha_0^\perp,\beta\rangle^2}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}\sum_{\substack{\gamma\in\mathcal{H}\\ \gamma\neq\alpha_0,\beta}}C_\gamma d_{\alpha_0,\beta,\gamma}-\frac{3}{N}\sum_{\substack{\beta,\gamma\in\mathcal{H}\\ \beta\neq\gamma\neq\alpha_0\neq\beta}}\frac{\langle\beta,\gamma\rangle C_{\alpha_0}C_\beta C_\gamma d_{\alpha_0,\beta,\gamma}}{\langle\alpha_0^\perp,\beta\rangle\langle\alpha_0^\perp,\gamma\rangle}\xi_\beta\xi_\gamma\widetilde{Q}_{\alpha_0,\beta,\gamma}\\
=&3C_{\alpha_0}\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle|\beta|^2C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}.
\end{aligned}
$$

□

Lemma 5.12 $S_{18}=-6|\alpha_0|^2\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle|\beta|^2C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}.$

Proof. By using (3.1) and $|\alpha_0|^2\xi_\beta\partial_{\xi,\beta}+|\beta|^2\xi_{\alpha_0}\partial_{\xi,\alpha_0}-\langle\alpha_0,\beta\rangle(\xi_\beta\partial_{\xi,\alpha_0}+\xi_{\alpha_0}\partial_{\xi,\beta})=\langle\alpha_0^\perp,\beta\rangle^2\langle\xi,\partial_\xi\rangle$, we can easily shown

$$
\begin{aligned}
&\xi_\beta^3\partial^3_{\xi,\alpha_0}\widetilde{P}_0-3\xi_{\alpha_0}\xi_\beta^3\partial^2_{\xi,\alpha_0}\widetilde{P}_2^\beta+3\xi_{\alpha_0}^3\xi_\beta\partial^2_{\xi,\beta}\widetilde{P}_2^{\alpha_0}-\xi_{\alpha_0}^3\partial^3_{\xi,\beta}\widetilde{P}_0\\
&\quad=\langle\alpha_0^\perp,\beta\rangle^3(D_\theta^3+3\langle\xi,\partial_\xi\rangle D_\theta-8D_\theta)\widetilde{P}_0+6\langle\alpha_0^\perp,\beta\rangle|\alpha|^2|\beta|^2D_\theta\widetilde{P}_0.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
S_{18}=&-\left(\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^3}\right)\frac{(D_\theta^3+3\langle\xi,\partial_\xi\rangle D_\theta-8D_\theta)\widetilde{P}_0}{\xi_{\alpha_0}}-6\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}|\alpha_0|^2|\beta|^2\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}\\
=&-6|\alpha_0|^2\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle|\beta|^2C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}.
\end{aligned}
$$

□

Proof of Theorem 5.2.

By the above long discussion, the equality $S_1+S_2+S_3=0$ reduces to the equality $S_{17}+S_{18}=0$. By Lemma 5.11 and Lemma 5.12, we have

$$
0=S_{17}+S_{18}=3(C_{\alpha_0}-2|\alpha_0|^2)\sum_{\substack{\beta\in\mathcal{H}\\ \beta\neq\alpha_0}}\frac{\langle\alpha_0,\beta\rangle|\beta|^2C_\beta}{\langle\alpha_0^\perp,\beta\rangle^5}\frac{D_\theta\widetilde{P}_0}{\xi_{\alpha_0}}.
$$

Since $\widetilde{P}_0$ is not a polynomial in $\xi_1^2+\xi_2^2$, $D_\theta\widetilde{P}_0$ is not zero, and the theorem is proved. □

6 Possible deformation of root systems

In this section, we investigate what kind of $\mathcal{H}$ and C_α satisfy (4.5) and (5.2).

Let $\#\mathcal{H} = N$ and $\mathcal{H} = \{\alpha_1, \dots, \alpha_N\}$. For notational convenience, we define

$$\begin{aligned}
A_{ij} :=& \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i^\perp, \alpha_j \rangle^3} \quad (\text{if } i \neq j), & A_{ii} :=& 0 \quad (\text{if } i = j), & \mathcal{A} :=& (A_{ij})_{1 \le i,j \le N}, \\
B_{ij} :=& \frac{\langle \alpha_i, \alpha_j \rangle |\alpha_i|^2 |\alpha_j|^2}{\langle \alpha_i^\perp, \alpha_j \rangle^5} \quad (\text{if } i \neq j), & B_{ii} :=& 0 \quad (\text{if } i = j), & \mathcal{B} :=& (B_{ij})_{1 \le i,j \le N}, \\
C_i :=& C_{\alpha_i}, & \boldsymbol{v} :=& {}^t(C_1, \dots, C_N).
\end{aligned}$$

Then, (4.5) and (5.2) are equivalent to

$$\mathcal{A}\boldsymbol{v} = \boldsymbol{0}, \tag{6.1}$$

$$\operatorname{diag}(C_1 - 2|\alpha_1|^2, \dots, C_N - 2|\alpha_N|^2)\mathcal{B}\boldsymbol{v} = \boldsymbol{0}, \tag{6.2}$$

respectively.

If N is odd, then $\det \mathcal{A} = 0$, since $\mathcal{A}$ is an alternative matrix. Therefore, the solution space of (6.1) is at least one dimensional. If the rank of $\mathcal{A}$ is $N-1$, the non-trivial solution of (6.1) is given by

$$C_i = C(-1)^i \mathrm{Pf}_i(\mathcal{A}) \qquad (i = 1, 2, \dots, N), \tag{6.3}$$

where C is a non-zero constant and $\mathrm{Pf}_i(\mathcal{A})$ is the Pfaffian of the $(N-1) \times (N-1)$ alternative matrix obtained by deleting the i th row and column of $\mathcal{A}$. Especially, if $N = 3$, then

$$C_1 = CA_{23}, \qquad C_2 = CA_{31}, \qquad C_3 = CA_{12}. \tag{6.4}$$

Lemma 6.1 *If $N = 3$ and $C_1 \neq 2|\alpha_1|^2$, α_2 and α_3 are symmetric with respect to the reflection r_{α_1}.*

Proof. Since $C_{\alpha_1} \neq 2|\alpha_1|^2$, the equations (6.2) and (6.4) imply

$$\begin{aligned}
B_{12}C_2 + B_{13}C_3 = 0 \quad &\Leftrightarrow \quad -C\frac{\langle \alpha_1, \alpha_3 \rangle \langle \alpha_1, \alpha_2 \rangle |\alpha_2|^2}{\langle \alpha_1^\perp, \alpha_3 \rangle^3 \langle \alpha_1^\perp, \alpha_2 \rangle^5} + C\frac{\langle \alpha_1, \alpha_2 \rangle \langle \alpha_1, \alpha_3 \rangle |\alpha_3|^2}{\langle \alpha_1^\perp, \alpha_2 \rangle^3 \langle \alpha_1^\perp, \alpha_3 \rangle^5} = 0 \\
&\Leftrightarrow \quad \langle \alpha_1^\perp, \alpha_3 \rangle^2 |\alpha_2|^2 = \langle \alpha_1^\perp, \alpha_2 \rangle^3 |\alpha_3|^2 \\
&\Leftrightarrow \quad \left(\frac{\langle \alpha_1^\perp, \alpha_3 \rangle}{|\alpha_1||\alpha_3|} \right)^2 = \left(\frac{\langle \alpha_1^\perp, \alpha_2 \rangle}{|\alpha_1||\alpha_2|} \right)^2 \\
&\Leftrightarrow \quad \sin^2\theta_2 = \sin^2\theta_3,
\end{aligned}$$

where θ_i are the angles from α_1 to α_i $(i = 2, 3)$. Since α_2 and α_3 are not parallel, this implies the lemma. $\square$

Corollary 6.2 *If $\#\mathcal{H} = 3$ and (i) more than or equal to two of C_i's are not equal to $2|\alpha_i|^2$ or (ii) all C_i's are equal to $2|\alpha_i|^2$, then $\mathcal{H}$ is a positive system of the A_2 type root system and L is the A_2 type CMS operator.*

PROOF. The first assertion follows directly from the last lemma.

If $C_i = 2|\alpha_i|^2$ for $i = 1, 2, 3$, (6.4) implies

$$\begin{aligned}
&\frac{\langle \alpha_2, \alpha_3 \rangle}{\langle \alpha_2^{\perp}, \alpha_3 \rangle^3 |\alpha_1|^2} = \frac{\langle \alpha_3, \alpha_1 \rangle}{\langle \alpha_3^{\perp}, \alpha_1 \rangle^3 |\alpha_2|^2} = \frac{\langle \alpha_1, \alpha_2 \rangle}{\langle \alpha_1^{\perp}, \alpha_2 \rangle^3 |\alpha_3|^2} \\
\Leftrightarrow \quad & \cot(\theta_3 - \theta_2)\{1 + \cot^2(\theta_3 - \theta_2)\} = -\cot\theta_3(1 + \cot^2\theta_3) = \cot\theta_2(1 + \cot^2\theta_2) \\
\Leftrightarrow \quad & \cot(\theta_3 - \theta_2) = -\cot\theta_3 = \cot\theta_2 \\
\Leftrightarrow \quad & \cot\theta_2 = -\cot\theta_3 = \pm 1/\sqrt{3}.
\end{aligned}$$

By changing the norm of vectors if necessary, we may regard $\mathcal{H}$ to be a positive system of A_2 type root system. □

As a result of Lemma 6.1 and Corollary 6.2, we obtain the following theorem.

Theorem 6.3 *When $\#\mathcal{H} = 3$, the possible hypeplane arrangement $\mathcal{H}$ is $\mathcal{H} = \{e_1, \pm a e_1 + e_2\}$ ($a \neq 0$). If $a \neq \pm 1/\sqrt{3}$, in other words if $\mathcal{H}$ is not a positive system of A_2 type, the coupling constants for $\pm a e_1 + e_2$ must be one.*

Next, let us consider the case $N = 4$.

Proposition 6.4 *If $\#\mathcal{H} = 4$, at least two vectors in $\mathcal{H}$ cross at right angles.*

PROOF. Put $\mathcal{H} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and let $x_i = \cot\theta_i$ ($i = 2, 3, 4$), where θ_i is the angle from α_1 to α_i. Note that x_i ($i = 2, 3, 4$) are all different since any two vectors in $\mathcal{H}$ are not parallel. Since

$$\begin{aligned}
A_{ij} &= \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i^{\perp}, \alpha_j \rangle^3} = \frac{1}{|\alpha_i|^2|\alpha_j|^2} \frac{\cos(\theta_j - \theta_i)}{\sin^3(\theta_j - \theta_i)} = \frac{1}{|\alpha_i|^2|\alpha_j|^2} \cot(\theta_j - \theta_i)(1 + \cot^2(\theta_j - \theta_i)), \\
B_{ij} &= \frac{\langle \alpha_i, \alpha_j \rangle |\alpha_i|^2 |\alpha_j|^2}{\langle \alpha_i^{\perp}, \alpha_j \rangle^5} = \frac{1}{|\alpha_i|^2|\alpha_j|^2} \frac{\cos(\theta_j - \theta_i)}{\sin^5(\theta_j - \theta_i)} = \frac{1}{|\alpha_i|^2|\alpha_j|^2} \cot(\theta_j - \theta_i)(1 + \cot^2(\theta_j - \theta_i))^2,
\end{aligned}$$

we have

$$\begin{aligned}
A_{1i} &= \frac{x_i(1 + x_i^2)}{|\alpha_1|^2|\alpha_i|^2}, & B_{1i} &= \frac{x_i(1 + x_i^2)^2}{|\alpha_1|^2|\alpha_i|^2} & &(i = 2, 3, 4), \\
A_{ij} &= \frac{(1 + x_i^2)(1 + x_j^2)(1 + x_i x_j)}{|\alpha_i|^2|\alpha_j|^2(x_i - x_j)^3}, & B_{ij} &= \frac{(1 + x_i^2)^2(1 + x_j^2)^2(1 + x_i x_j)}{|\alpha_i|^2|\alpha_j|^2(x_i - x_j)^5} & &(2 \leq i \neq j \leq 4).
\end{aligned}$$

Assume that any two vectors in $\mathcal{H}$ do not cross at right angles and deduce contradiction. Note that $x_i \neq 0$ for $i = 2, 3, 4$ and $1 + x_i x_j \neq 0$ for $2 \leq i \neq j \leq 4$ since $\langle \alpha_i, \alpha_j \rangle \neq 0$ for any i, j.

We divide the proof of this theorem into three parts, since the method of poof is different for the following cases:

(1) More than or equal to three coupling constants are one.

(2) More than or equal to three coupling constants are not one.

(3) Just two coupling constants are one.

(1) In this case, we may assume $C_i = 2|\alpha_i|^2$ for $i = 2, 3, 4$. Let $p_k := \sum_{i=2}^{4} x_i^k$ be the k-th power sum of x_2, x_3, x_4. By (6.1) and (6.2), we have

$$\begin{cases} 2(A_{12}|\alpha_2|^2 + A_{13}|\alpha_3|^2 + A_{14}|\alpha_4|^2) = 0 \\ 2(B_{12}|\alpha_2|^2 + B_{13}|\alpha_3|^2 + B_{14}|\alpha_4|^2) = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} p_1 + p_3 = 0 \\ p_1 + 2p_3 + p_5 = 0 \end{cases} \quad \Leftrightarrow \quad p_1 = -p_3 = p_5,$$

Since $6p_5 = p_1^5 - 5p_1^3p_2 + 5p_1^2p_3 + 5p_2p_3$, we have

$$\begin{aligned}
p_1\{5(p_1^2+1)p_2 - (p_1^4 - 5p_1^2 - 6)\} = 0 \quad &\Leftrightarrow \quad p_1(p_1^2+1)(5p_2 - p_1^2 + 6) = 0 \\
&\Leftrightarrow \quad p_1(p_1^2+1)\left(3\sum_{i=2}^{4} x_i^2 + \sum_{2\le i<j\le 4}(x_i - x_j)^2 + 6\right) = 0 \\
&\Rightarrow \quad p_1 = 0.
\end{aligned}$$

If $p_1 = 0$, then $p_3 = x_2^3 + x_3^3 - (x_2+x_3)^3 = 3x_2x_3x_4 = 0$, which contradicts our assumption.
(2) Assume that the coupling constants for $\alpha_1, \alpha_2, \alpha_3$ are not one. Then $C_1, \dots, C_4$ satisfy

$$\begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} \\ B_{21} & 0 & B_{23} & B_{24} \\ B_{31} & B_{32} & 0 & B_{34} \\ A_{41} & A_{42} & A_{43} & 0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \mathbf{0}. \tag{6.5}$$

Since all C_i's are not zero, the determinant of the coefficient matrix is zero;

$$(B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23})(B_{12}A_{34} - B_{13}A_{24} + B_{23}A_{14}) = 0. \tag{6.6}$$

On the other hand, since $C_1, \dots, C_4$ satisfy (6.1), we have

$$\mathrm{Pf}(\mathcal{A}) = A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0.$$

By our assumption that any two vectors in $\mathcal{H}$ do not cross at right angles, each A_{ij} is not zero. Therefore, the solution of (6.1) is expressed as

$$C_1 = sA_{34}, \qquad C_2 = tA_{34}, \qquad C_3 = -(sA_{14} + tA_{24}), \qquad C_4 = sA_{13} + tA_{23}, \qquad (s, t \neq 0).$$

Since this satisfies (6.5), we have

$$s(A_{13}B_{14} - A_{14}B_{13}) + t(B_{12}A_{34} - B_{13}A_{24} + B_{23}A_{14}) = 0.$$

Assume that $B_{12}A_{34} - B_{13}A_{24} + B_{23}A_{14} = 0$. In this case,

$$A_{13}B_{14} = A_{14}B_{13} \quad \Leftrightarrow \quad x_3(1+x_3^2)x_4(1+x_4^2)^2 = x_4(1+x_4^2)x_3(1+x_3^2)^2$$

and this implies $x_3 = -x_4$, since $x_3, x_4 \neq 0$ and $x_3 \neq x_4$. Therefore, $A_{14}|\alpha_4|^2 = -A_{13}|\alpha_3|^2$, $B_{14}|\alpha_4|^2 = -B_{13}|\alpha_3|^2$, and we have

$$\begin{cases} A_{12}C_2 + A_{13}C_3 + A_{14}C_4 = 0 \\ B_{12}C_2 + B_{13}C_3 + B_{14}C_4 = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} A_{12}C_2 + A_{13}(C_3 - |\alpha_3|^2C_4/|\alpha_4|^2) = 0 \\ B_{12}C_2 + B_{13}(C_3 - |\alpha_3|^2C_4/|\alpha_4|^2) = 0. \end{cases}$$

By these equations, we have $(B_{13}A_{12} - B_{12}A_{13})C_2 = 0$, which implies $x_2^2 = x_3^2$. Since $x_2 \neq x_3$, we have $x_2 = -x_3 = x_4$. But this contradicts the condition $x_2 \neq x_4$.

Therefore, $B_{12}A_{34} - B_{13}A_{24} + B_{23}A_{14} \neq 0$, and A_{ij}, B_{ij} satisfy

$$\begin{cases} A_{12}A_{34} - A_{13}A_{24} + A_{14}A_{23} = 0 \\ B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \frac{x_2(1+x_3x_4)}{(x_3-x_4)^3} + \frac{x_3(1+x_4x_2)}{(x_4-x_2)^3} + \frac{x_4(1+x_2x_3)}{(x_2-x_3)^3} = 0 \\ \frac{x_2(1+x_3x_4)}{(x_3-x_4)^5} + \frac{x_3(1+x_4x_2)}{(x_4-x_2)^5} + \frac{x_4(1+x_2x_3)}{(x_2-x_3)^5} = 0, \end{cases}$$

by (6.6). From these equations, we have

$$\frac{x_2(1+x_3x_4)(2x_3-x_2-x_4)}{(x_3-x_4)^6}=\frac{x_3(1+x_2x_4)(2x_2-x_3-x_4)}{(x_2-x_4)^6} \tag{6.7}$$

$$\begin{aligned}\Leftrightarrow\quad & x_2(1+x_3x_4)(2x_3-x_2-x_4)(2x_4-x_2-x_3)\\ &\qquad \times\{(x_2-x_4)^4+(x_2-x_4)^2(x_3-x_4)^2+(x_3-x_4)^4\}\\ &\quad +(x_2+x_3+x_4+3x_2x_3x_4)(x_3-x_4)^6=0.\end{aligned} \tag{6.8}$$

Here, we used $x_2 \neq x_3$. Similarly, we have

$$\begin{aligned}& x_2(1+x_3x_4)(2x_4-x_2-x_3)(2x_3-x_2-x_4)\\ &\qquad \times\{(x_2-x_3)^4+(x_2-x_3)^2(x_3-x_4)^2+(x_3-x_4)^4\}\\ &\quad +(x_2+x_3+x_4+3x_2x_3x_4)(x_3-x_4)^6=0.\end{aligned} \tag{6.9}$$

By (6.8) and (6.9), we have

$$\begin{aligned}& x_2(1+x_3x_4)(2x_2-x_3-x_4)(2x_3-x_4-x_2)(2x_4-x_2-x_3)\\ &\qquad \times\{(x_2-x_4)^2+(x_2-x_3)^2+(x_3-x_4)^2\}=0.\end{aligned}$$

Since $x_2 \neq 0$, $1+x_3x_4 \neq 0$ and x_2, x_3, x_4 are all different, $2x_2 = x_3 + x_4$, $2x_3 = x_4 + x_2$ or $2x_4 = x_2 + x_3$. But this is impossible since we obtain $x_2 = x_3$ or $x_4(1 + x_2x_3) = 0$ from (6.7).
(3) Let us assume the coupling constants for α_3 and α_4 are 1 and those for α_1 and α_2 are not 1. In this case, all the conditions for A_{ij}, B_{ij} and C_i are

$$\begin{aligned}& A_{12}C_2+2|\alpha_3|^2A_{13}+2|\alpha_4|^2A_{14}=0, && B_{12}C_2+2|\alpha_3|^2B_{13}+2|\alpha_4|^2B_{14}=0,\\ & A_{21}C_1+2|\alpha_3|^2A_{23}+2|\alpha_4|^2A_{24}=0, && B_{21}C_1+2|\alpha_3|^2B_{23}+2|\alpha_4|^2B_{24}=0,\\ & \mathrm{Pf}(A)=A_{12}A_{34}-A_{13}A_{24}+A_{14}A_{23}=0. &&\end{aligned}$$

By eliminating C_1 and C_2, we have

$$x_2^2=\frac{x_3^2-x_3x_4+x_4^2+x_3^4-x_3^3x_4+x_3^2x_4^2-x_3x_4^3+x_4^4}{1+x_3^2-x_3x_4+x_4^2}, \tag{6.10}$$

$$\frac{(1+x_3^2)(1+x_2x_3)(1+2x_2x_3-x_2^2)}{(x_2-x_3)^5}+\frac{(1+x_4^2)(1+x_2x_4)(1+2x_2x_4-x_2^2)}{(x_2-x_4)^5}=0, \tag{6.11}$$

$$\frac{x_2(1+x_3x_4)}{(x_3-x_4)^3}+\frac{x_3(1+x_4x_2)}{(x_4-x_2)^3}+\frac{x_4(1+x_2x_3)}{(x_2-x_3)^3}=0. \tag{6.12}$$

Let $F(x_2)$ and $G(x_2)$ be the left hand sides of (6.11) and (6.12), respectively. Since $\{F(x_2)G(-x_2)-F(-x_2)G(x_2)\}/x_2$ is an even polynomial in x_2, it is a polynomial in x_2^2. By substituting (6.10) for it and putting $x_3 = s + t$, $x_4 = s - t$, we have

$$\begin{aligned}& t^6(s^2-t^2)\\ &\ \times(3s^2+24s^4+75s^6+120s^8+105s^{10}+48s^{12}+9s^{14}+3t^2+92s^2t^2+559s^4t^2+1408s^6t^2\\ &\qquad +1745s^8t^2+1060s^{10}t^2+253s^{12}t^2+44t^4+889s^2t^4+4368s^4t^4+8658s^6t^4+7580s^8t^4\\ &\qquad +2445s^{10}t^4+237t^6+3744s^2t^6+14538s^4t^6+20616s^6t^6+9777s^8t^6+600t^8+7461s^2t^8\\ &\qquad +20616s^4t^8+15675s^6t^8+773t^{10}+6932s^2t^{10}+10263s^4t^{10}+492t^{12}+2415s^2t^{12}+123t^{14})\\ &\ \times(5s^4+30s^6+75s^8+100s^{10}+75s^{12}+30s^{14}+5s^{16}+50s^2t^2+478s^4t^2+1744s^6t^2+3196s^8t^2\end{aligned}$$

$$
\begin{aligned}
&+3154s^{10}t^2+1606s^{12}t^2+332s^{14}t^2+121t^4+1818s^2t^4+9358s^4t^4+22792s^6t^4+28609s^8t^4\\
&+17910s^{10}t^4+4432s^{12}t^4+1386t^6+14280s^2t^6+52248s^4t^6+87916s^6t^6+69246s^8t^6\\
&+20684s^{10}t^6+6543t^8+48852s^2t^8+123037s^4t^8+126170s^6t^8+45458s^8t^8+16172t^{10}\\
&+83490s^2t^{10}+126546s^4t^{10}+54532s^6t^{10}+21879t^{12}+69266s^2t^{12}+45080s^4t^{12}+15210t^{14}\\
&+21860s^2t^{14}+4225t^{16})\\
=\;&0.
\end{aligned}
$$

This equation implies $t=0$, $s=\pm t$ or $(s,t)=(0,0)$, but these are impossible since $x_3, x_4 \neq 0$ and $x_3 \neq x_4$. $\square$

Theorem 6.5 *When $\#\mathcal{H}=4$, the possible singular locus $\mathcal{H}$ is $\mathcal{H}=\{e_1, e_2, \pm ae_1+e_2\}$ ($a\neq 0$).*

Proof. By Proposition 6.4, at least two vectors in $\mathcal{H}$, say α_1 and α_2, cross at right angles. In this case, we have $A_{12}=0$ and we may assume $\alpha_2=\alpha_1^\perp$. Note that $\alpha_2^\perp=\alpha_1^{\perp\perp}=-\alpha_1$. Then,

$$
\begin{aligned}
\mathrm{Pf}(\mathcal{A})=-A_{13}A_{24}+A_{14}A_{23}=0 \quad &\Leftrightarrow \quad \frac{\langle\alpha_1,\alpha_3\rangle}{\langle\alpha_1^\perp,\alpha_3\rangle^3}\frac{\langle\alpha_2,\alpha_4\rangle}{\langle\alpha_2^\perp,\alpha_4\rangle^3}=\frac{\langle\alpha_1,\alpha_4\rangle}{\langle\alpha_1^\perp,\alpha_4\rangle^3}\frac{\langle\alpha_2,\alpha_3\rangle}{\langle\alpha_2^\perp,\alpha_3\rangle^3}\\
&\Leftrightarrow \quad \frac{\langle\alpha_1,\alpha_3\rangle}{\langle\alpha_1^\perp,\alpha_3\rangle^3}(-1)\frac{\langle\alpha_1^\perp,\alpha_4\rangle}{\langle\alpha_1,\alpha_4\rangle^3}=\frac{\langle\alpha_1,\alpha_4\rangle}{\langle\alpha_1^\perp,\alpha_4\rangle^3}(-1)\frac{\langle\alpha_1^\perp,\alpha_3\rangle}{\langle\alpha_1,\alpha_3\rangle^3}\\
&\Leftrightarrow \quad \left(\frac{\langle\alpha_1,\alpha_3\rangle}{\langle\alpha_1^\perp,\alpha_3\rangle}\right)^4=\left(\frac{\langle\alpha_1,\alpha_4\rangle}{\langle\alpha_1^\perp,\alpha_4\rangle}\right)^4\\
&\Leftrightarrow \quad x_3^4=x_4^4\\
&\Leftrightarrow \quad x_3=-x_4,
\end{aligned}
$$

since $x_3\neq x_4$. By an appropriate coordinate change, we may assume $\alpha_1=e_1$. Then $\alpha_2=e_1^\perp=e_2$ and, by changing the norm of α_3,α_4 if necessary, we have $\alpha_3=x_3e_1+e_2$ and $\alpha_4=-x_3e_1+e_2$. $\square$

7 New deformation of B_2 type commutative pair

In this section, we construct a pair of commuting differential operators with the hyperplane arrangement $\mathcal{H}=\{\alpha_1:=e_1, \alpha_2:=e_2, \alpha_\pm:=\pm ae_1+e_2\}$. If $a=\pm1$, then $\mathcal{H}$ is the positive system of B_2 type root system and the commuting differential operators are known. Therefore, we assume $a\neq\pm1$ in this section.

If the coupling constants for $\alpha_\pm$ are one, the existence of such commuting operators is proved by Veselov-Feigin-Chalykh for rational or trigonometric potential cases ([2]). Here, we consider the case where the coupling constants for $\alpha_\pm$ are two. In this case, there exists a pair of commuting differential operators L and P for rational, trigonometric and even elliptic cases. Remember the lowest order of the commutant P for the original CMS model is four. But in our case, we can not find a fourth order commutant P because no fourth order operator satisfies Proposition 3.2. The lowest order of a commutant P is six.

Let the coupling constants for $\alpha_\pm$ be two and let $C_\pm := C_{\alpha_\pm}=2\cdot(2+1)|\alpha_\pm|^2=6(a^2+1)$. Since C_1, C_2 satisfy (6.1), (6.2), we have

$$
\begin{aligned}
&aC_1+\frac{1}{a^3}C_2-\frac{1-a^2}{8a^3}6(a^2+1)=0, \qquad aC_1+\frac{1}{a^5}C_2-\frac{(1-a^2)(a^2+1)}{32a^5}6(a^2+1)=0\\
\Leftrightarrow \qquad &C_1=\frac{3}{16a^2}(a^2+1)(3a^{-2}-1), \qquad C_2=\frac{3}{16}(a^2+1)(3a^2-1).
\end{aligned}
$$

By these equations, $a \neq \pm\sqrt{3}, \pm 1/\sqrt{3}$, since $C_1, C_2 \neq 0$.

For $\alpha \in \mathcal{H}$, let u_α be a function of the form

$$u_\alpha = u_\alpha(x_\alpha), \qquad u_\alpha(t) = \frac{C_\alpha}{t^2} + (\text{real analytic at } t = 0),$$

and consider the equation $[L, P] = 0$ for

$$L = -(\partial_{x_1}^2 + \partial_{x_2}^2) + \sum_{\alpha \in \mathcal{H}} u_\alpha.$$

If a is generic, C_1, C_2 are not of the form $k(k+1)|\alpha_i|^2$ ($k \in \boldsymbol{Z}$) for $i = 1, 2$. Therefore, the principal symbol $\widetilde{P}_0$ of P is an even polynomial in ξ_1 and ξ_2 because of Proposition 3.2 (1). Moreover, since the coupling constants for $\alpha_\pm$ are two, $\widetilde{P}_0$ must satisfy $\lim_{\xi_{\alpha_\pm} \to 0} \partial_{\xi, \alpha_\pm} \widetilde{P}_0 = \lim_{\xi_{\alpha_\pm} \to 0} \partial^3_{\xi, \alpha_\pm} \widetilde{P}_0 = 0$ because of Proposition 3.2 (2). Such a polynomial of degree six is unique up to constant multiple and modulo $(\xi_1^2 + \xi_2^2)^3$. We choose

$$\widetilde{P}_0 = a(4 - a^2)\xi_1^6 + 5a\xi_1^4\xi_2^2 + 5a^{-1}\xi_1^2\xi_2^4 + a^{-1}(4 - a^{-2})\xi_2^6.$$

Proposition 7.1 *If $a^2 \neq 7/3, 3/7, (13 \pm 4\sqrt{10})/3$, then we have*

$$u_1(t) = \frac{3(3a^{-2} - 1)(a^2 + 1)}{4}\wp(2at), \quad u_2(t) = \frac{3(3a^2 - 1)(a^2 + 1)}{4}\wp(2t), \quad u_\pm(t) = 6(a^2 + 1)\wp(t),$$

modulo constant factors.

Sketch of proof. Let us consider the equation (4.3). Even if the potential functions u_α are not rational, we can show that $F(x, \xi)$ in (4.3) is a polynomial in ξ of degree two. Since $[\partial_{x_2}\partial_{\xi_1} - \partial_{x_1}\partial_{\xi_2}, \langle \xi, \partial_x \rangle] = 0$, we have

$$(\partial_{x_2}\partial_{\xi_1} - \partial_{x_1}\partial_{\xi_2})^3 \left(\sum_{\substack{\alpha, \beta \in \mathcal{H} \\ \alpha \neq \beta}} u'_\alpha u_\beta \partial_{\xi, \alpha} \widetilde{P}_2^\beta \right) = 0. \tag{7.1}$$

Here, we used $\partial^3_{\xi, \alpha^\perp} \partial_{\xi, \alpha} \widetilde{P}_2^\alpha = 0$ for any $\alpha \in \mathcal{H}$, which is a direct consequence of $\lim_{\xi_\alpha \to 0} \partial^3_{\xi, \alpha} \widetilde{P}_0 = 0$.

Choose $\alpha_0 \in \mathcal{H}$. By taking the limit $\lim_{x_{\alpha_0} \to 0} x_{\alpha_0}^6 \times (7.1)$, we obtain

$$\sum_{\substack{\beta \in \mathcal{H} \\ \beta \neq \alpha_0}} (\partial^3_{\xi, \alpha_0^\perp} \partial_{\xi, \alpha_0} \widetilde{P}_2^\beta) u_\beta(\langle \alpha_0^\perp, \beta \rangle t) = 0,$$

Similarly, we obtain

$$\sum_{\substack{\beta \in \mathcal{H} \\ \beta \neq \alpha_0}} \partial_{\xi, \alpha_0^\perp} (\partial^3_{\xi, \alpha_0} \widetilde{P}_2^\beta - C_{\alpha_0} \partial_{\xi, \alpha_0} \widetilde{P}_4^{\alpha_0, \beta}) u_\beta(\langle \alpha_0^\perp, \beta \rangle t) = 0$$

from (5.1). These equations imply

$$\begin{aligned}
&(3a^{-2} - 7)\{u_+(t) - u_-(t)\} = 0, \\
&(3a^2 - 7)\{u_+(-at) - u_-(at)\} = 0, \\
&2a^2 u_1(-t) - 2u_2(at) + (a^2 - 1)u_-(2at) = 0, \\
&2a^2 u_1(-t) - 2u_2(-at) + (a^2 - 1)u_+(-2at) = 0, \\
&3a(a^2 - 1)(a^2 + 1)^2(3a^4 - 26a^2 + 3)\{8a^2 u_1(t) + (a^2 - 3)u_+(2at)\} = 0.
\end{aligned}$$

Here, we have abbreviated u_{α_1} to u_1 etc. If $a^2 \neq 7/3, 3/7, (13 \pm 4\sqrt{10})/3$, we obtain from these that $u_+(t)$ is an even function, $u_1(t) = (3a^{-2} - 1)u_+(2at)/8$, $u_2(t) = (3a^2 - 1)u_+(2t)/8$ and $u_-(t) = u_+(t)$.

Finally, we can show $u_+(t) = 6(a^2 + 1)\wp(t)$ by the same method as in §7 of [3], namely, by studying the coefficients in the Laurent expansion of (7.1) as a function of x_{α_+}. □

For such potential function, we can construct a commutant P of L. Since we can check the commutativity by a direct method, we omit the proof and write the conclusion only.

Before the statement of theorem, we introduce some notation. Let g_2, g_3 be the invariants of $\wp$ appearing in the differential equation $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$. As above, we abbreviate u_{α_1} to u_1 etc. We put

$$L_1 := \partial_{x_1}^2 - u_1, \qquad L_2 := \partial_{x_2}^2 - u_2, \qquad L_\pm := \partial_{x,\alpha_\pm}^2 - (a^2+1)u_\pm$$

and

$$A_\pm(5) := \partial_{x,\alpha_\pm}^5 - \frac{5}{2}u_\pm\partial_{x,\alpha_\pm}^3 - \frac{15}{4}u_\pm'\partial_{x,\alpha_\pm}^2 + \frac{1}{8}(a^2+1)^2\{15u_\pm^2 - 25(a^2+1)u_\pm''\}\partial_{x,\alpha_\pm}.$$

Note that $-(\partial_{x_1}^2 + \partial_{x_2}^2) + u_\pm$ commutes with $L_\pm$, since $\partial_{x_1}^2 + \partial_{x_2}^2 = (a^2+1)^{-1}(\partial_{x,\alpha_\pm}^2 + \partial_{x,\alpha_\pm^\perp}^2)$. Moreover, since the coupling constants for $\alpha_\pm$ are two, $L_\pm$ has a commutant of order five ([1]). The operator $A_\pm(5) - (21/8)(a^2+1)^4 g_2\partial_{x,\alpha_\pm}$ is such a commutant.

Theorem 7.2 *Let*

$$L = -(\partial_{x_1}^2 + \partial_{x_2}^2) + u_1 + u_2 + u_+ + u_-.$$

Then the following operator commutes with L.

$$\begin{aligned} P =& a(4-a^2)L_1^3 + 5aL_1^2L_2 + 5a^{-1}L_1L_2^2 + a^{-1}(4-a^{-2})L_2^3 + P_2 \\ &+ P_4 + \frac{1}{2}\langle\partial_x, \partial_\xi\rangle\ \widetilde{P}_4\Big|_{\xi\to\partial_x} + \frac{1}{8}\langle\partial_x, \partial_\xi\rangle^2\widetilde{P}_4 + Q_4 + P_6, \end{aligned}$$

where

$$\begin{aligned} P_2 :=& \frac{1}{(a^2+1)^4}[-20a(a^2+1)(u_+\partial_{x,\alpha_+^\perp}^4 + u_-\partial_{x,\alpha_-^\perp}^4) \\ &- 10a^{-1}(a^{-2} - 4 + a^2)\{(L_+^2 - \partial_{x,\alpha_+}^4)\partial_{x,\alpha_+^\perp}^2 + (L_-^2 - \partial_{x,\alpha_-}^4)\partial_{x,\alpha_-^\perp}^2\} \\ &- 2(a^2-1)(3a^{-2}-1)(3a^2-1)\{(A_+(5) - \partial_{x,\alpha_+}^5)\partial_{x,\alpha_+^\perp} - (A_-(5) - \partial_{x,\alpha_-}^5)\partial_{x,\alpha_-^\perp}\} \\ &- a(a^4 - 6a^2 + 6 - 6a^{-2} + a^{-4})\{(L_+^3 - \partial_{x,\alpha_+}^6) + (L_-^3 - \partial_{x,\alpha_-}^6)\}], \\ P_4 =& u_1(u_+ + u_-)\{6a(4-a^2)\partial_{x_1}^2 + 7(a + a^{-1})\partial_{x_2}^2\} + u_1(u_+ - u_-)(3a^2 - 7)\partial_{x_1}\partial_{x_2} \\ &+ u_2(u_+ + u_-)\{7(a + a^{-1})\partial_{x_1}^2 + 6a^{-1}(4 - a^{-2})\partial_{x_2}^2\} + u_2(u_+ - u_-)(3a^{-2} - 7)\partial_{x_1}\partial_{x_2} \\ &+ \frac{1}{8}(u_+ + u_-)\{(35a^{-1} + 156a - 39a^3)\partial_{x_1}^2 + (35a + 156a^{-1} - 39a^{-3})\partial_{x_2}^2\}, \\ Q_4 =& -\frac{21}{16}(a^2+1)^2(a - a^{-1})(3a^2-1)(3a^{-2}-1)g_2\left(L_1 - L_2 - \frac{a^2-1}{a^2+1}(u_+ + u_-)\right) \end{aligned}$$

and

$$
\begin{aligned}
P_6 =& 3a(4-a^2)(u_1''-u_1^2)(u_++u_-)+3a^{-1}(4-a^{-2})(u_2''-u_2^2)(u_++u_-)\\
&+\frac{1}{8}\{(15a^3-72a-7a^{-1})u_1+(15a^{-3}-72a^{-1}-7a)u_2\}(u_+^2+u_-^2)\\
&+\frac{a}{4}\{(20a^4-83a^2+24+7a^{-2})u_1+(20a^{-4}-83a^{-2}+24+7a^2)u_2\}(u_+''+u_-'')\\
&+\frac{3}{16}(a+a^{-1})(7a^2-38+7a^{-2})(u_+^2u_-+u_+u_-^2)-\frac{5}{32}a(5a^4-202+5a^{-4})(u_+''u_-+u_+u_-'')\\
&+\frac{3}{16}a(a^2-a^{-2})(19a^2-122+19a^{-2})u_+'u_-'-7(a+a^{-1})u_1u_2(u_++u_-)\\
&+\frac{1}{8}\{a(57a^2-216+7a^{-2})u_1+a^{-1}(57a^{-2}-216+7a^2)u_2\}u_+u_-.
\end{aligned}
$$

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